

# Vorticity

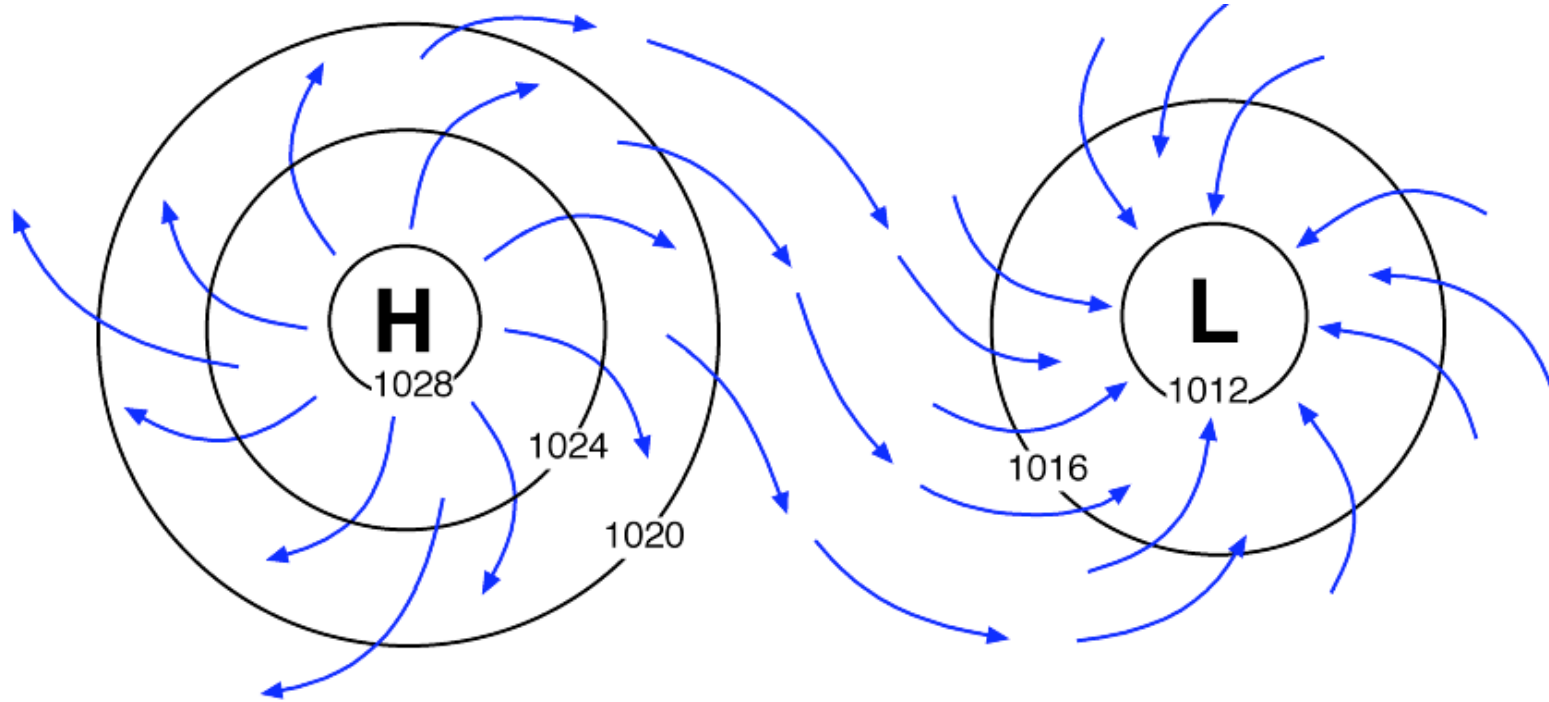
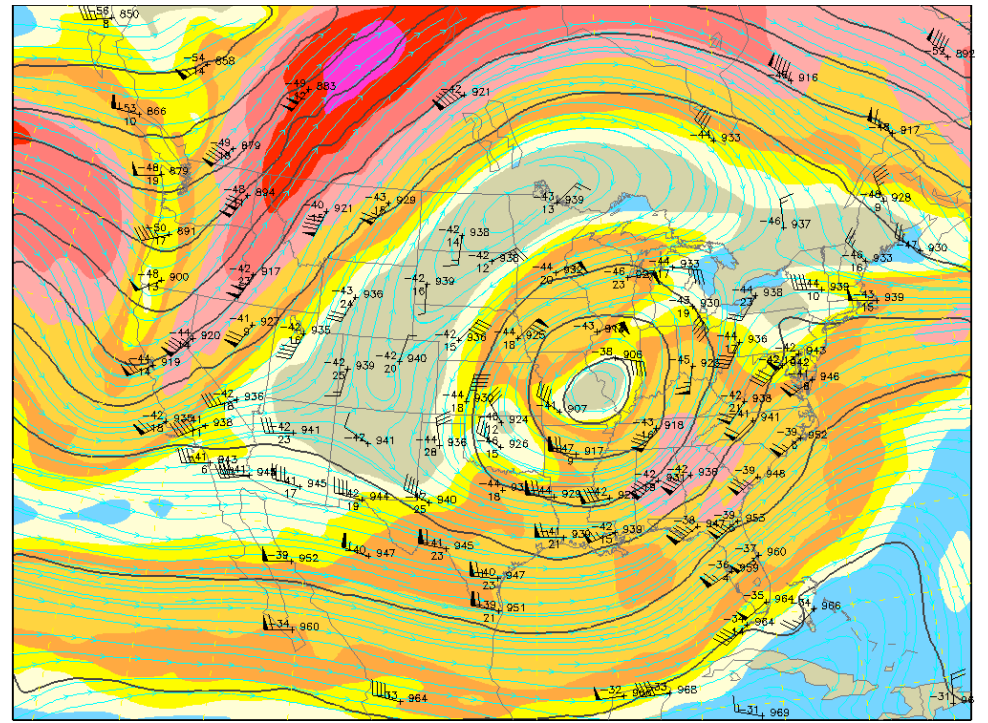
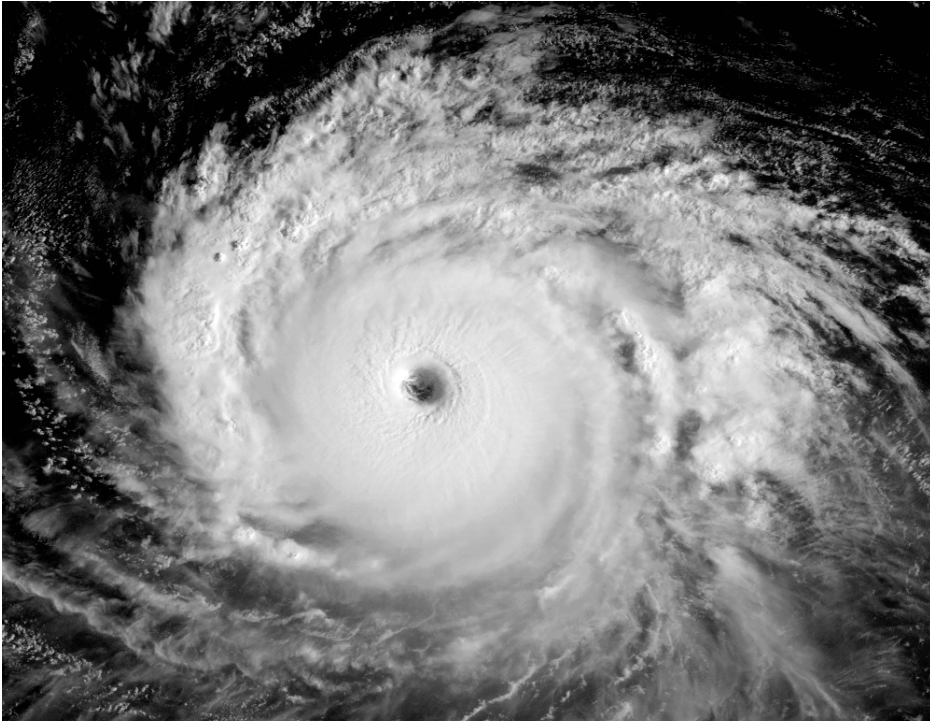
- As noted last week, the vorticity is a vector representing the microscopic (individual air parcels) rotation of a fluid and is defined as the curl ( $\vec{\nabla} \times$ ) of the velocity:

$$\vec{\omega} = \vec{\nabla} \times \vec{V}$$

which in Cartesian coordinates is

$$\vec{\omega} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \hat{i} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - \hat{j} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \hat{k} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

- As should be evident from the weather maps and synoptic scale circulation systems (hurricanes, troughs, lows, etc.) we've looked at in class, dynamic meteorology is mainly concerned with the vertical,  $\hat{k}$ , component of the vorticity (i.e., rotation about the vertical axis).



- With this in mind, we define

$$\vec{\eta} = \hat{k} \cdot (\vec{\nabla} \times \vec{V}_A) \quad \text{and} \quad \vec{\zeta} = \hat{k} \cdot (\vec{\nabla} \times \vec{V})$$

as the vertical components of the absolute vorticity and the relative vorticity, respectively, where  $\vec{V}_A = \vec{V} + \vec{\Omega} \times \vec{R}$  is the absolute velocity, i.e., the Earth relative velocity plus the velocity due to the rotation of the Earth.

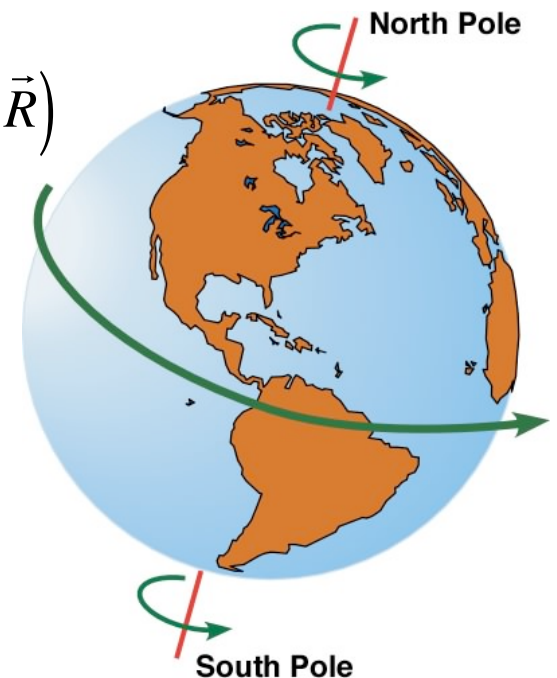
- Thus, the difference between the absolute and relative vorticities can be seen as:

$$\begin{aligned} \vec{\eta} &= \hat{k} \cdot (\vec{\nabla} \times \vec{V}_A) = \hat{k} \cdot (\vec{\nabla} \times (\vec{V} + \vec{\Omega} \times \vec{R})) = \hat{k} \cdot (\vec{\nabla} \times \vec{V} + \vec{\nabla} \times \vec{\Omega} \times \vec{R}) \\ \vec{\eta} &= \hat{k} \cdot (\vec{\nabla} \times \vec{V}) + \hat{k} \cdot [\vec{\Omega}(\vec{\nabla} \cdot \vec{R}) - \vec{R}(\vec{\nabla} \cdot \vec{\Omega})] \end{aligned}$$

- Where we've used the vector identity

$$\vec{a} \times \vec{b} \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

and the fact that the rotation rate does not change with position on the Earth!



- Noting  $\vec{R} = (x, y, 0)$  and expanding out the second term on the right side:

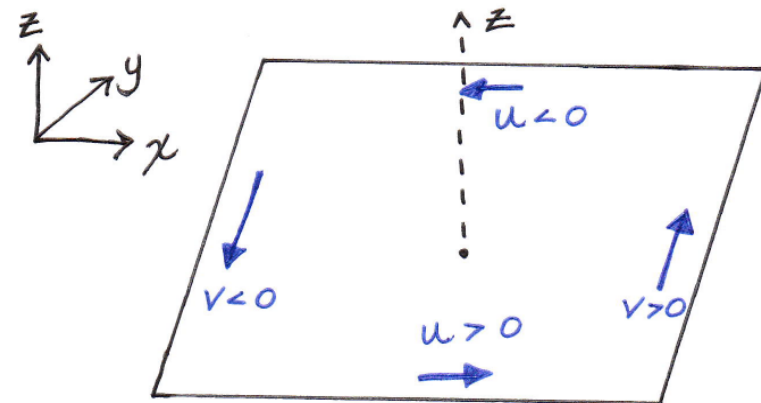
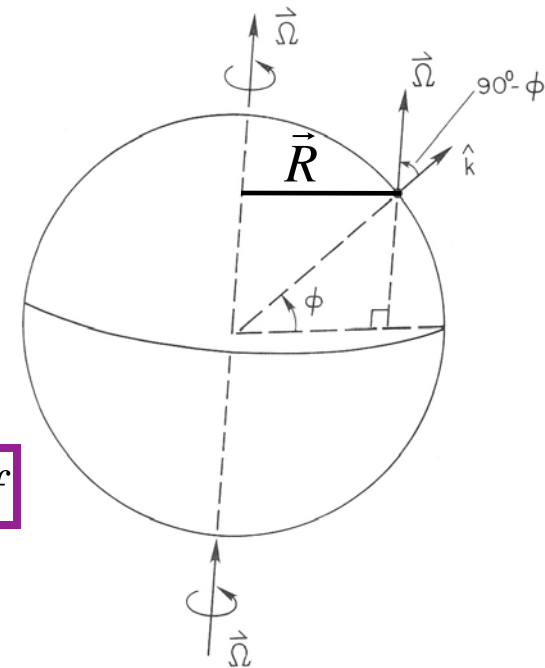
$$\vec{\eta} = \vec{\zeta} + \hat{k} \cdot \left[ \vec{\Omega} \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + 0 \right) \right] = \vec{\zeta} + \hat{k} \cdot 2\vec{\Omega}$$

and the component of  $\vec{\Omega}$  in the  $\hat{k}$  direction is  $\Omega \sin \phi$  so  $\vec{\eta} = \vec{\zeta} + 2\Omega \sin \phi = \vec{\zeta} + f$

- Thus, we find that

$$\vec{\eta} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \quad \text{and} \quad \vec{\zeta} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

and just like the circulation, positive vorticity is counterclockwise (cyclonic) and negative vorticity is clockwise (anticyclonic) as illustrated in the figure (in the Northern Hemisphere).

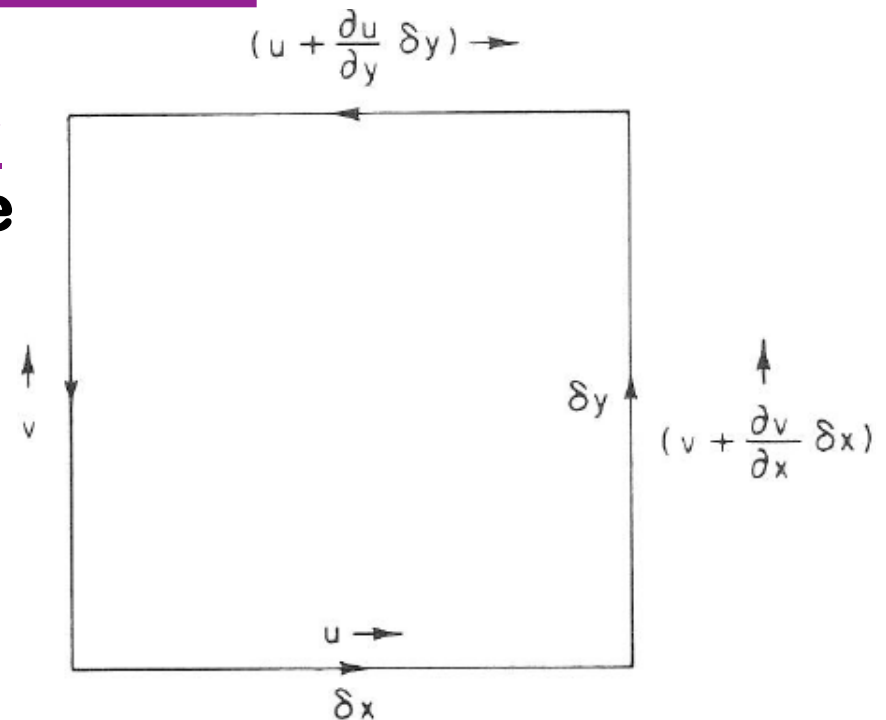


\*  $\frac{\partial v}{\partial x} > 0$  and  $\frac{\partial u}{\partial y} < 0$   
 \* So  $\zeta > 0$   
 CYCLONIC!

- The connection between vorticity and circulation can be illustrated by defining the vertical component of vorticity to be the circulation around a closed path in a horizontal plane divided by the area of fluid enclosed by the path, in the limit where the area goes to 0:

$$\vec{\zeta} \equiv \lim_{A \rightarrow 0} \oint \frac{\vec{V} \cdot d\vec{r}}{A} = \hat{k} \cdot (\vec{\nabla} \times \vec{V})$$

- The equivalence of the two definitions of relative vorticity can be seen by calculating the circulation around the rectangular area  $\delta x \delta y$  shown in the figure to the right.



- Summing up the  $\vec{V} \cdot d\vec{r}$  for each side of the rectangle finds:

$$C = \oint u dx + v dy = u \delta x + \left( v + \frac{\partial v}{\partial x} \delta x \right) \delta y - \left( u + \frac{\partial u}{\partial y} \delta y \right) \delta x - v \delta y$$

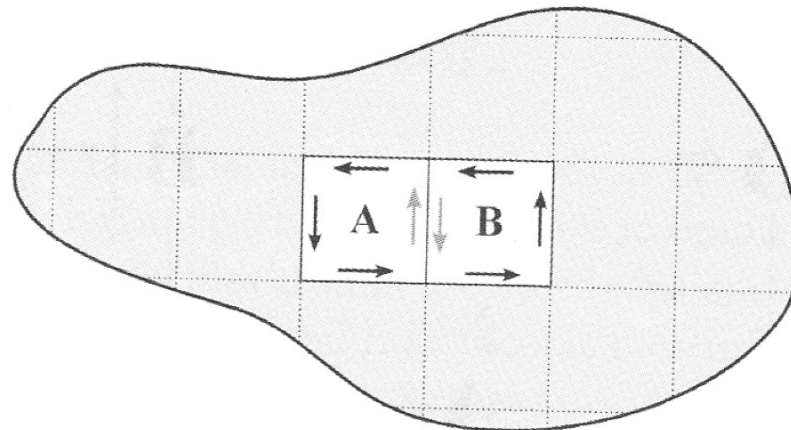
$$C = \oint u dx + v dy = u \delta x + \left( v + \frac{\partial v}{\partial x} \delta x \right) \delta y - \left( u + \frac{\partial u}{\partial y} \delta y \right) \delta x - v \delta y = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y$$

- Dividing by the area,  $\delta x \delta y$ , of the rectangle:

$$\frac{C}{A} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \equiv \xi$$

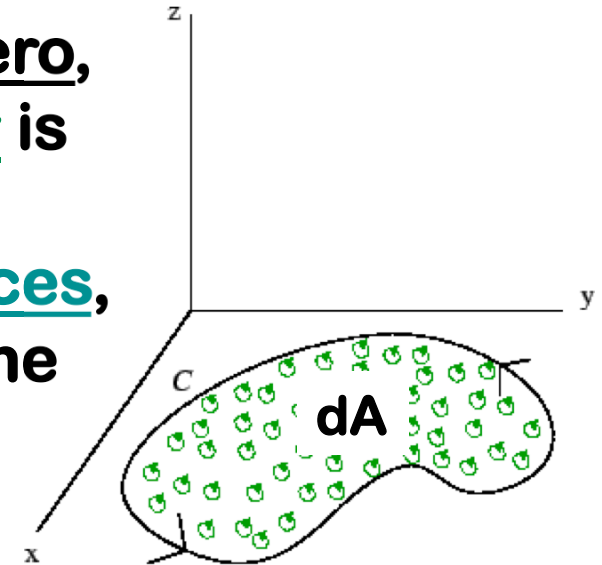
and the definitions are equivalent!

- To further illustrate the relationship between circulation and vorticity, we divide a closed fluid loop into a large number of tiny squares and calculate the circulation about each square, as illustrated below.



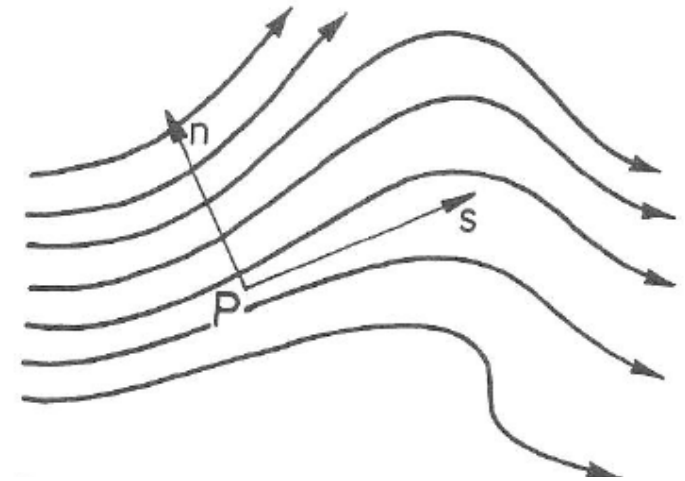
- Because of the cancellation of the circulations about common sides, only the contributions from the outer edges, i.e., those sides of squares that run along the contour of the closed fluid loop, need to be considered.
- If we shrink the size of the squares to zero, the total circulation around the contour is the sum of the circulations of each infinitesimally small square (point vortices, measured by vorticity) added up over the area the contour encloses:

$$\oint u dx + v dy = \iint_A \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y$$



- Or, more generally, we may write:  $\oint \vec{V} \cdot d\vec{r} = \iint_A (\vec{\nabla} \times \vec{V}) \cdot \hat{n} dA$  which states that the circulation about a closed path is equal to the component of vorticity normal to the area, dA, enclosed by the path, or simply Stoke's Theorem applied to the velocity!

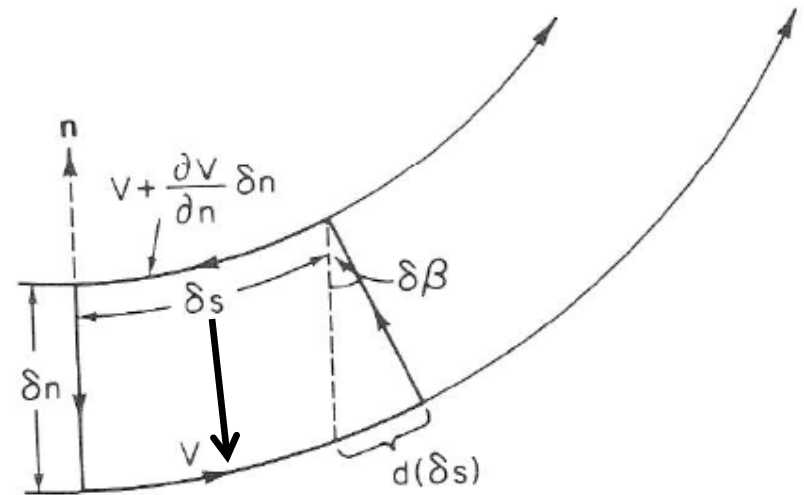
- Switching gears, because vorticity is associated with curvature of the height and wind fields, the physical interpretation of vorticity is best understood using natural coordinates.



- To determine the equation for vorticity in natural coordinates we consult the figure to the right and evaluate the circulation between the contours:

$$\delta C = V[\delta s + d(\delta s)] - \left( V + \frac{\partial V}{\partial n} \delta n \right) \delta s$$

$$\delta C = V\delta s + Vd(\delta s) - V\delta s - \frac{\partial V}{\partial n} \delta n \delta s$$



- We note that  $d(\delta s) = \delta\beta\delta n$  where  $\delta\beta$  is the change in the wind direction over the distance  $\delta s$ .

- Therefore,

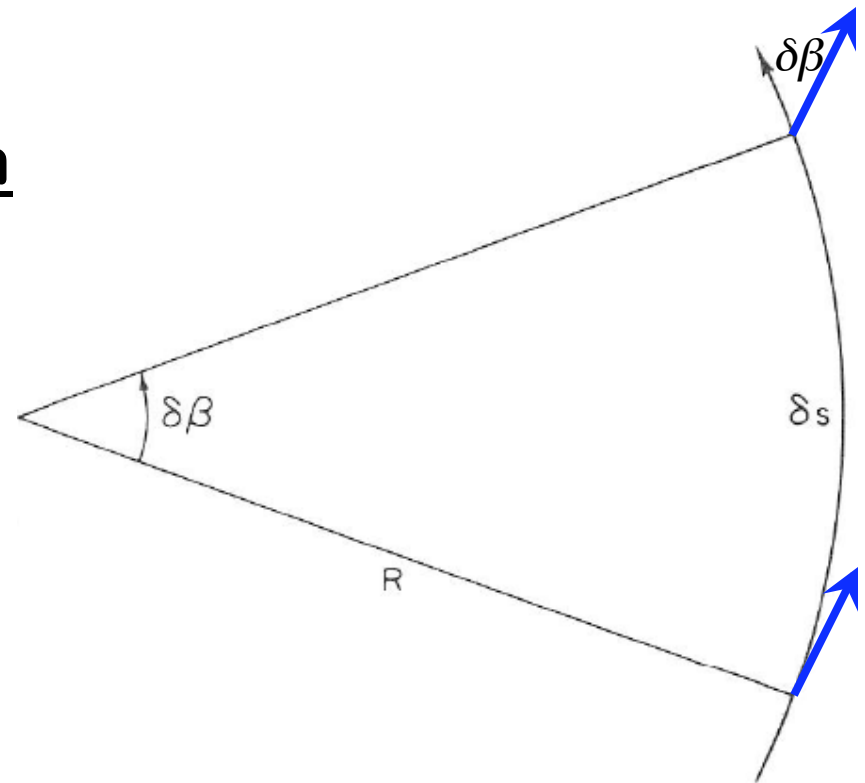
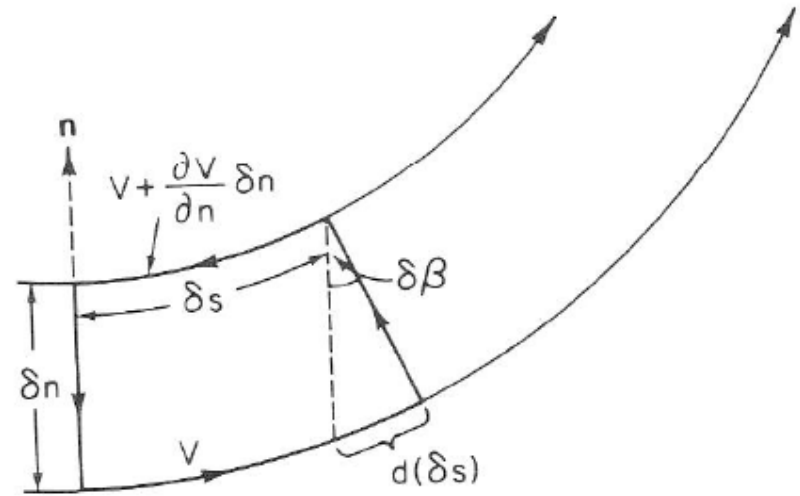
$$\delta C = V\delta\beta\delta n - \frac{\partial V}{\partial n}\delta n\delta s$$

$$\delta C = \left( -\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s} \right)\delta n\delta s$$

- In the limit of the area between the contours going to 0:

$$\vec{\zeta} = \lim_{\delta n, \delta s \rightarrow 0} \frac{\delta C}{\delta n\delta s} = -\frac{\partial V}{\partial n} + \frac{V}{R}$$

where **R** is the radius of curvature of the contours following the **curved flow** (streamlines) as shown to the right.



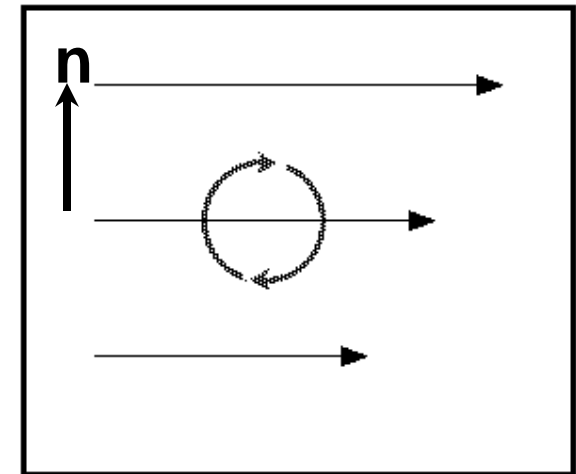
• Thus, in natural coordinates the vorticity has two terms:

1) The change in the wind speed normal to the direction of the flow,  $-dV/dn$ , called the shear vorticity.

2) The speed of the wind divided by the radius of curvature,  $V/R$ , representing the turning of the wind (recall it came from  $\delta\beta/\delta s$ ) following the flow, called the curvature vorticity.

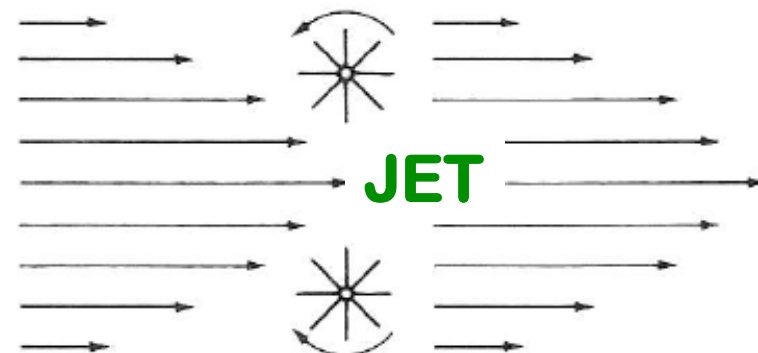
• Writing the relative vorticity in this way illustrates the important fact that one does not need curved flow to have vorticity. All one needs is a change in the wind speed normal to the flow as seen to the right.

• Here,  $dV/dn > 0$  ( $\hat{n}$  is + to the left of motion) and  $\zeta < 0$ , clockwise rotation.



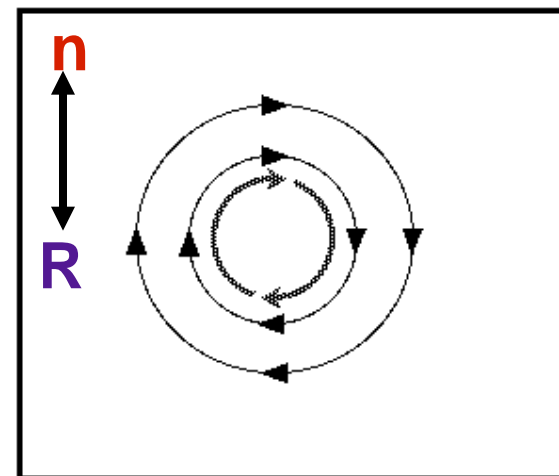
Shear Vorticity

- In the atmosphere, we also see the case to the right quite often to the north and south of the jet stream (see next slides!).



- Here,  $\frac{dV}{dn} < 0$  north and  $\frac{dV}{dn} > 0$  south of the wind speed maximum, and thus we have a vorticity couplet or dipole with a counterclockwise (cyclonic) circulation north and a clockwise (anticyclonic) circulation south of the jet.

- The curvature vorticity term can be illustrated with the figure to the right where  $\frac{V}{R}$  is  $< 0$  because  $V$  is always  $> 0$  and  $R$  is  $< 0$  because it's in the  $-\hat{n}$  direction. So  $\zeta < 0$ , and the flow is anticyclonic.

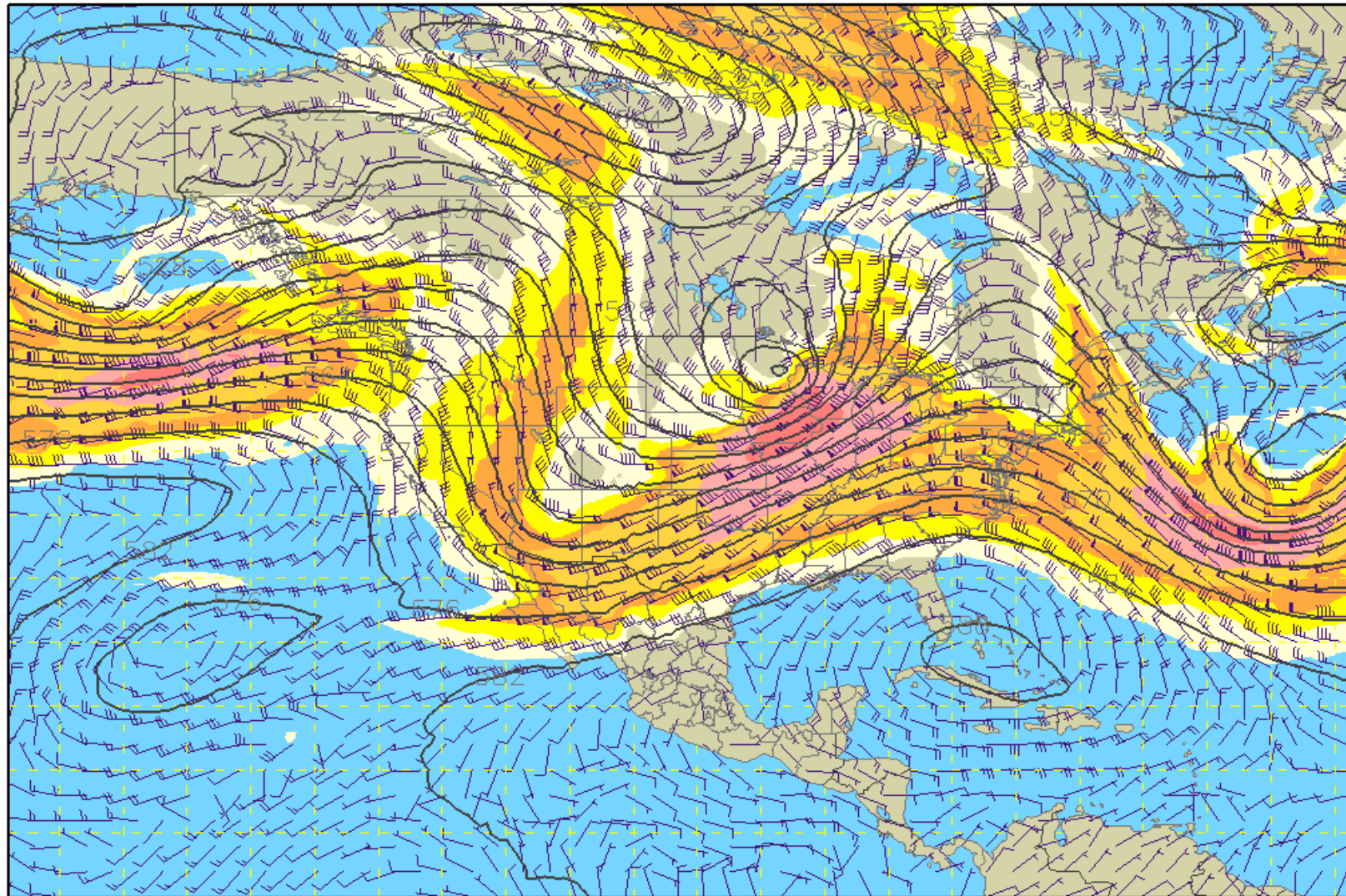


Curvature Vorticity

# 500 mb Heights (dm) / Isotachs (knots)

84-hour forecast valid 1200 UTC Thu 25 Nov 2010

GFS (00z 22 Nov)

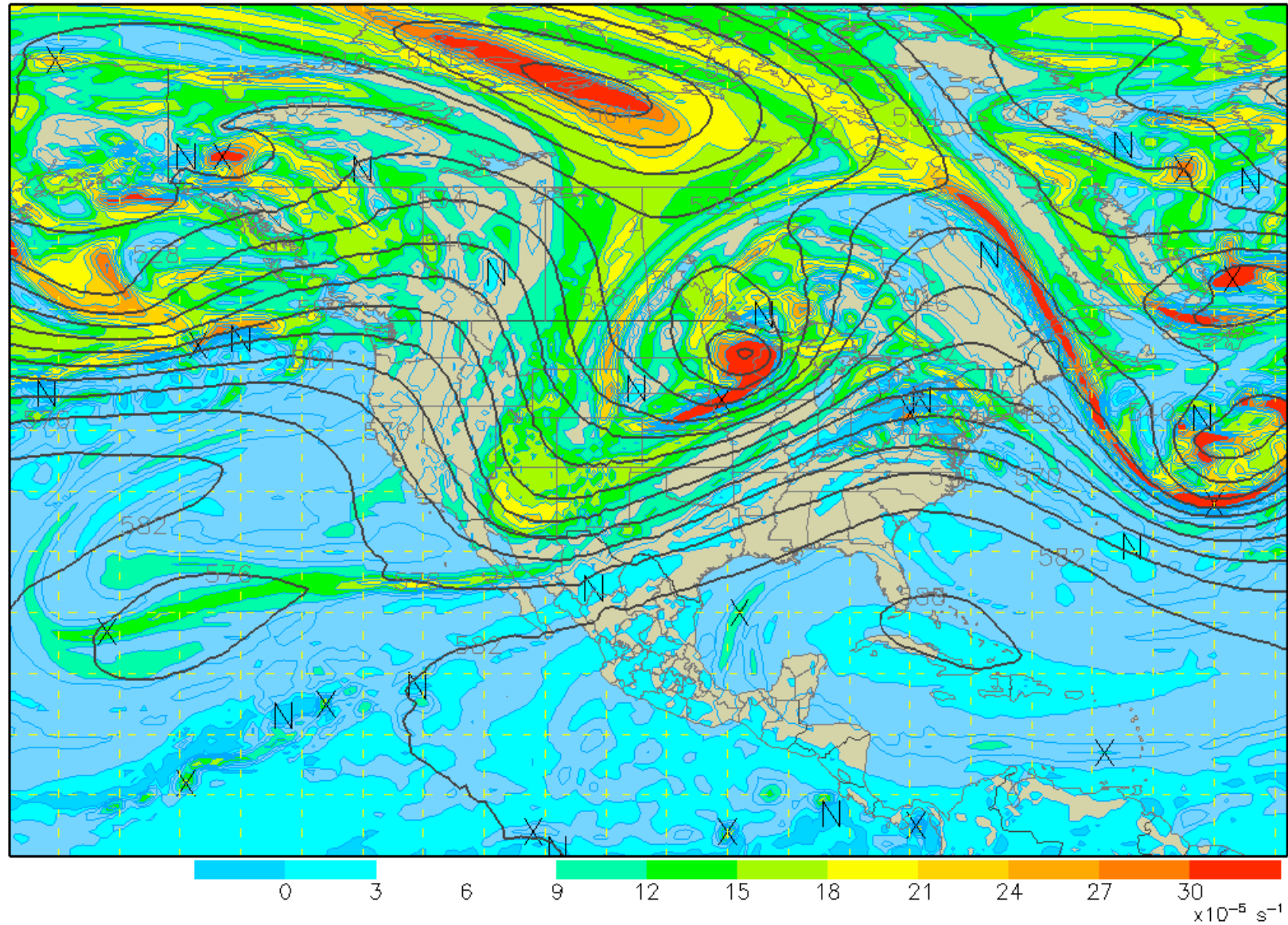


30 40 50 60 80 100 125 150 knots

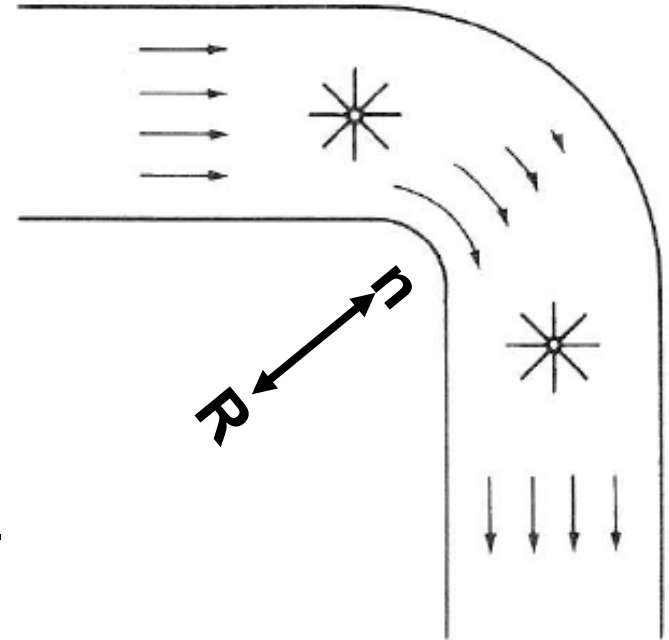
# 500 mb Heights (dm) / Abs. Vorticity ( $\times 10^{-5} \text{ s}^{-1}$ )

84-hour forecast valid 1200 UTC Thu 25 Nov 2010

GFS (00z 22 Nov)



- Interestingly, because the relative vorticity is the sum of the shear and curvature terms, curved flow can actually have zero vorticity if the two terms offset one another, as shown in the example on the right.



- As the flow turns the corner,  $\underline{V/R} < \underline{0}$  because  $\underline{R}$  is in the  $\underline{-\hat{n}}$  direction.
- Thus, for the vorticity to be zero,  $\underline{-dV/dn}$  has to be positive and  $\underline{dV/dn} < \underline{0}$ , so that  $\underline{V}$  must decrease (slow down) in the  $\underline{+\hat{n}}$  direction as shown in the figure.
- $\underline{R}$  increases in the  $\underline{+\hat{n}}$  direction at a constant rate, so  $\underline{V/R}$  will decrease and  $\underline{-dV/dn}$  must have an equal and opposite constant value; thus as the speed decreases on the outside corner of the pipe, the speed must increase on the inner side of the pipe.