

The Vorticity Equation

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$$\oint \frac{D}{Dt} (\vec{V} \cdot d\vec{r}) \equiv \frac{DC}{Dt} = -\oint \frac{\vec{\nabla} p}{\rho} \cdot d\vec{r} - \oint (2\vec{\Omega} \times \vec{V}) \cdot d\vec{r}$$

which relates the time rate of change of the circulation to two physically interpretable terms (the pressure gradient and Coriolis forces), we can derive an expression, the Vorticity Equation, relating the rate of change of vorticity to several physical processes in the atmosphere.

- To accomplish this goal, we start with the frictionless Momentum Equation:

$$\frac{D\vec{V}}{Dt} = -\frac{\vec{\nabla} p}{\rho} - f\hat{k} \times \vec{V}$$

and apply the $\hat{k} \cdot \vec{\nabla} \times$ operator to both sides. Why?

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Because the vertical component of vorticity is $\vec{\zeta} \equiv \hat{k} \cdot \vec{\nabla} \times \vec{V}$!

- We first focus on the left hand side, $D\vec{V} / Dt$:

$$\hat{k} \cdot \vec{\nabla} \times \frac{D\vec{V}}{Dt} = \hat{k} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{Du}{Dt} & \frac{Dv}{Dt} & \frac{Dw}{Dt} \end{vmatrix} = \frac{\partial}{\partial x} \left(\frac{Dv}{Dt} \right) - \frac{\partial}{\partial y} \left(\frac{Du}{Dt} \right)$$

because the $\hat{k} \cdot$ means we only want the \hat{k} component!

- To see what these two terms really are, we replace the total derivatives (D/Dt ; Lagrangian) with the sum of the local (Eulerian) and advective terms:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

so that

$$\hat{k} \cdot \vec{\nabla} \times \frac{D\vec{V}}{Dt} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

- This is **pretty messy** as there are **14 terms** because **each** of the $\frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial x} \right)$ **terms turns into two terms**, $\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2}$, but by **interchanging the order of differentiation** and **cleverly collecting similar terms** by remembering we want an expression involving **vorticity**, we find that:

$$\hat{k} \cdot \vec{\nabla} \times \frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + v \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + w \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)$$

which **looks hideous**, but **some of the terms should also be familiar to you** (hopefully!).

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$$1: \frac{\partial \xi}{\partial t} \quad 2-4: \vec{V} \cdot \vec{\nabla} \xi \quad 5: \xi \quad 6: \vec{\nabla} \cdot \vec{V} \quad 7: \hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z}$$

- Therefore, the left hand side of our expression is

$$\hat{k} \cdot \vec{\nabla} \times \frac{D\vec{V}}{Dt} = \frac{\partial \vec{\zeta}}{\partial t} + \vec{V} \cdot \nabla \vec{\zeta} + \vec{\zeta} (\vec{\nabla} \cdot \vec{V}) + \hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z}$$

which has a lot of interesting terms, but we still need to deal with the right hand side of the [Momentum Equation!](#)

- Next, we look at the [pressure gradient term](#):

$$\hat{k} \cdot \vec{\nabla} \times \left(-\frac{\vec{\nabla} p}{\rho} \right) = \hat{k} \cdot \frac{\vec{\nabla} \rho \times \vec{\nabla} p}{\rho^2}$$

which looks very similar to our [pressure gradient term](#) from the [Circulation Theorem](#).

- And finally, we have the [Coriolis term](#) $\hat{k} \cdot \vec{\nabla} \times (-f\hat{k} \times \vec{V})$ for which we will do out the matrices on the next slide.

$$\hat{k} \cdot \vec{\nabla} \times (-f\hat{k} \times \vec{V}) = -\hat{k} \cdot \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & f \\ u & v & w \end{vmatrix} = -\hat{k} \cdot \vec{\nabla} \times [(-fv)\hat{i} - (-fu)\hat{j}]$$

$$-\hat{k} \cdot \vec{\nabla} \times [(-fv)\hat{i} + (fu)\hat{j}] = -\hat{k} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -fv & fu & 0 \end{vmatrix} = -\left[\frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial y}(-fv) \right]$$

$$-\left[\frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial y}(-fv) \right] = -u \frac{\partial f}{\partial x} - f \frac{\partial u}{\partial x} - v \frac{\partial f}{\partial y} - f \frac{\partial v}{\partial y}$$

- Now we note that **f** only depends on **y** (**latitude**) and **define** $\beta \equiv \frac{\partial f}{\partial y}$ so that we may write the **Coriolis term** as

$$\hat{k} \cdot \vec{\nabla} \times (-f\hat{k} \times \vec{V}) = -v\beta - f(\vec{\nabla} \cdot \vec{V}_h)$$

- **Collecting ALL of our terms from the left and right hand sides, we now have the following expression for the Vorticity Equation:**

$$\frac{\partial \vec{\zeta}}{\partial t} = \underbrace{-\vec{V} \cdot \nabla \vec{\zeta}}_1 - \underbrace{\vec{\zeta}(\vec{\nabla} \cdot \vec{V})}_2 - \underbrace{\hat{k} \cdot \vec{\nabla} w \times \frac{\partial \vec{V}}{\partial z}}_3 + \underbrace{\hat{k} \cdot \left(\frac{\vec{\nabla} \rho \times \vec{\nabla} p}{\rho^2} \right)}_4 - \underbrace{\nu \beta}_5 - \underbrace{f(\vec{\nabla} \cdot \vec{V})}_6$$

1

2

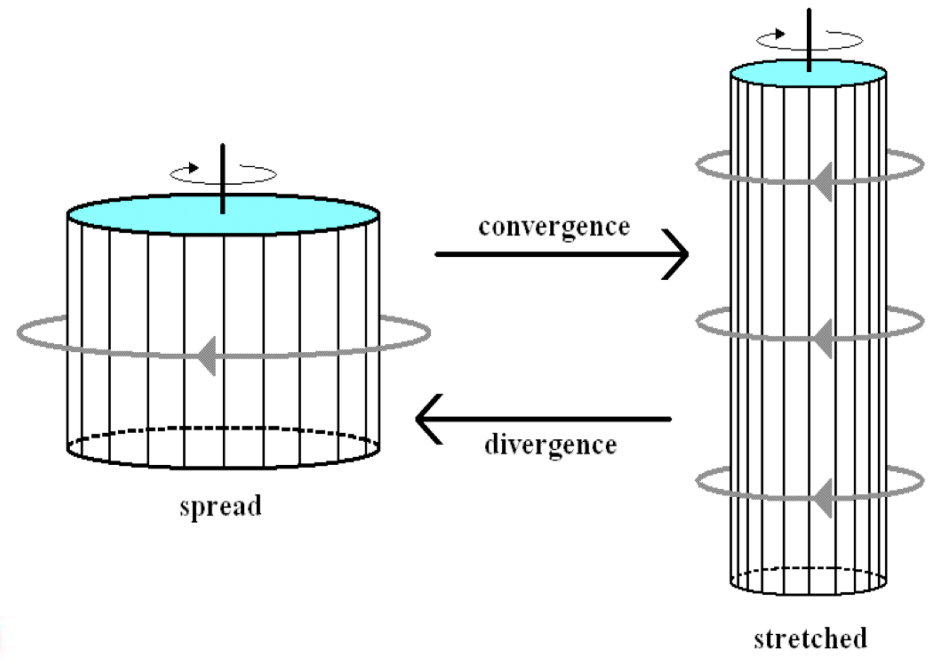
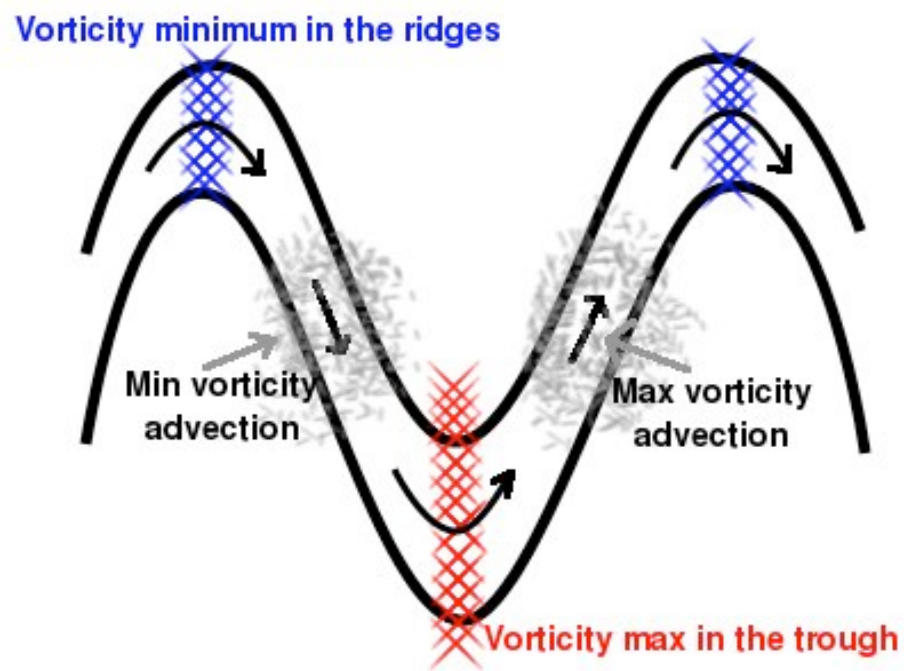
3

4

5

6

- The Vorticity Equation says that the local time rate of change of relative vorticity is related to SIX separate physical processes:



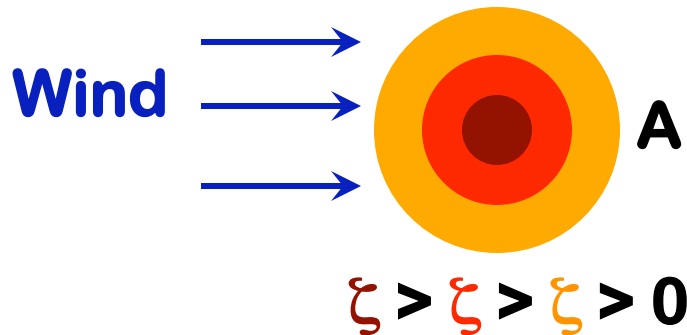
$$\frac{\partial \vec{\zeta}}{\partial t} = \underbrace{-\vec{V} \cdot \nabla \vec{\zeta}}_{1} - \underbrace{\vec{\zeta}(\vec{\nabla} \cdot \vec{V})}_{2} - \underbrace{\hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z}}_{3} + \underbrace{\hat{k} \cdot \left(\frac{\vec{\nabla} \rho \times \vec{\nabla} p}{\rho^2} \right)}_{4} - \underbrace{\nu \beta}_{5} - \underbrace{f(\vec{\nabla} \cdot \vec{V})}_{6}$$

- The Vorticity Equation says that the local time rate of change of relative vorticity is related to **SIX** separate physical processes:

- 1) The advection of relative vorticity.
- 2) The divergence acting on relative vorticity.
- 3) The tilting of horizontal shear vorticity.
- 4) Baroclinic (or solenoid) processes.
- 5) The advection of planetary vorticity.
- 6) The divergence acting on planetary vorticity.

1) The advection of relative vorticity: $-\vec{v} \cdot \nabla \zeta$

- We've already examined advection and this term simply represents the transport of relative vorticity by the flow.

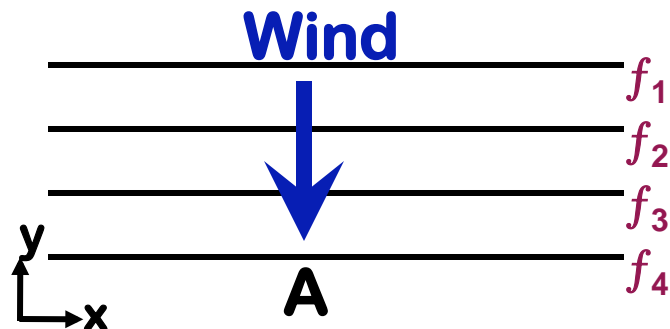


At point A, the gradient of ζ points to the left, so

$$-\vec{v} \cdot \nabla \zeta = -|\vec{v}| |\nabla \zeta| \cos 180^\circ > 0 \rightarrow \frac{\partial \zeta}{\partial t} > 0$$

5) The advection of planetary vorticity: $-v\beta$

- We recall that $\beta \equiv \frac{\partial f}{\partial y}$ and $f = 2\Omega \sin \phi$ so we have higher values of f to the north and lower to the south ($f_1 > f_4$).



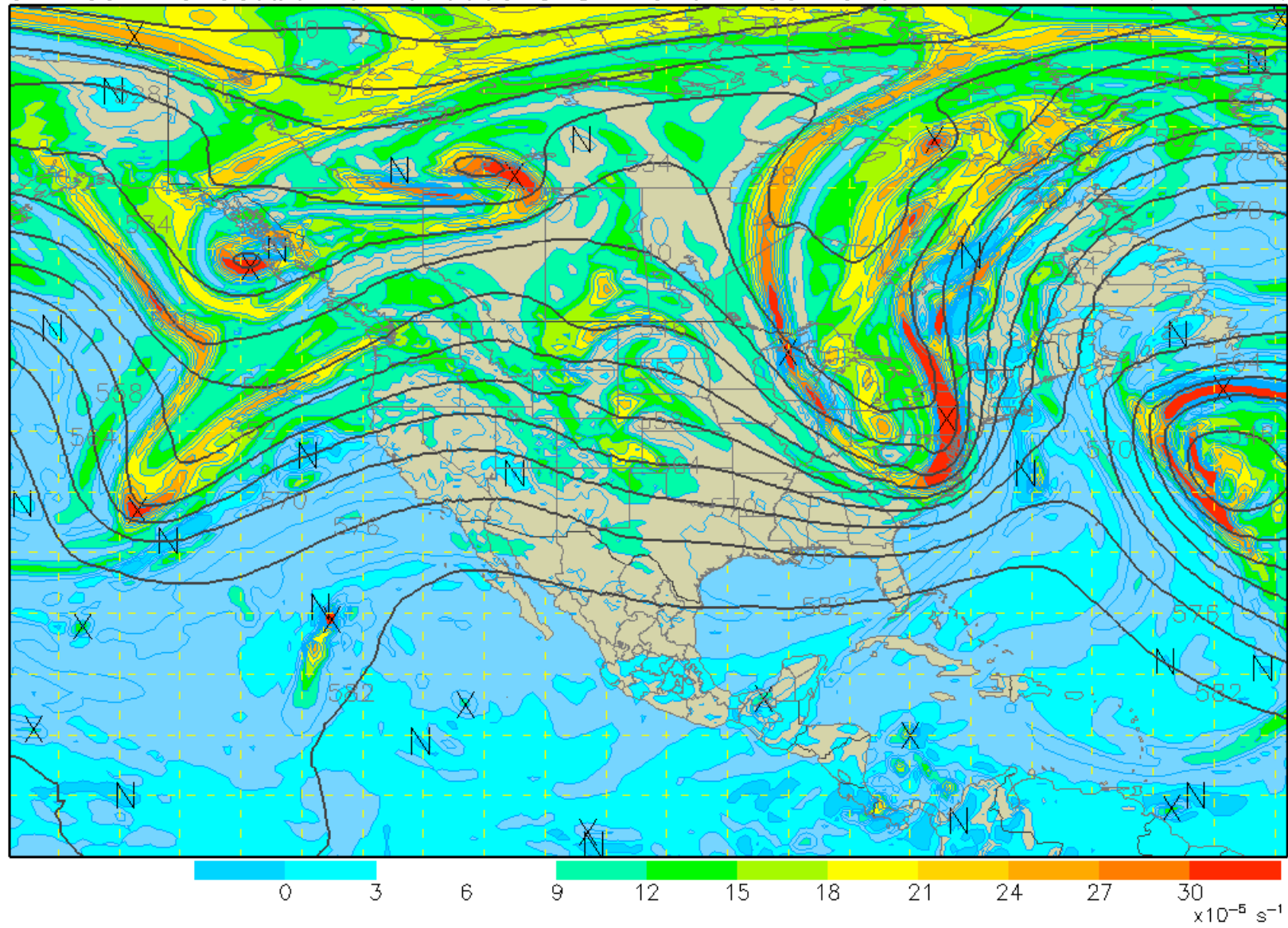
So a northerly wind ($\underline{v} < \underline{0}$) will advect higher f to the south, so

$$-v\beta > 0 \rightarrow \frac{\partial \zeta}{\partial t} > 0$$

500 mb Heights (dm) / Abs. Vorticity ($\times 10^{-5} \text{ s}^{-1}$)

84-hour forecast valid 0000 UTC Thu 02 Dec 2010

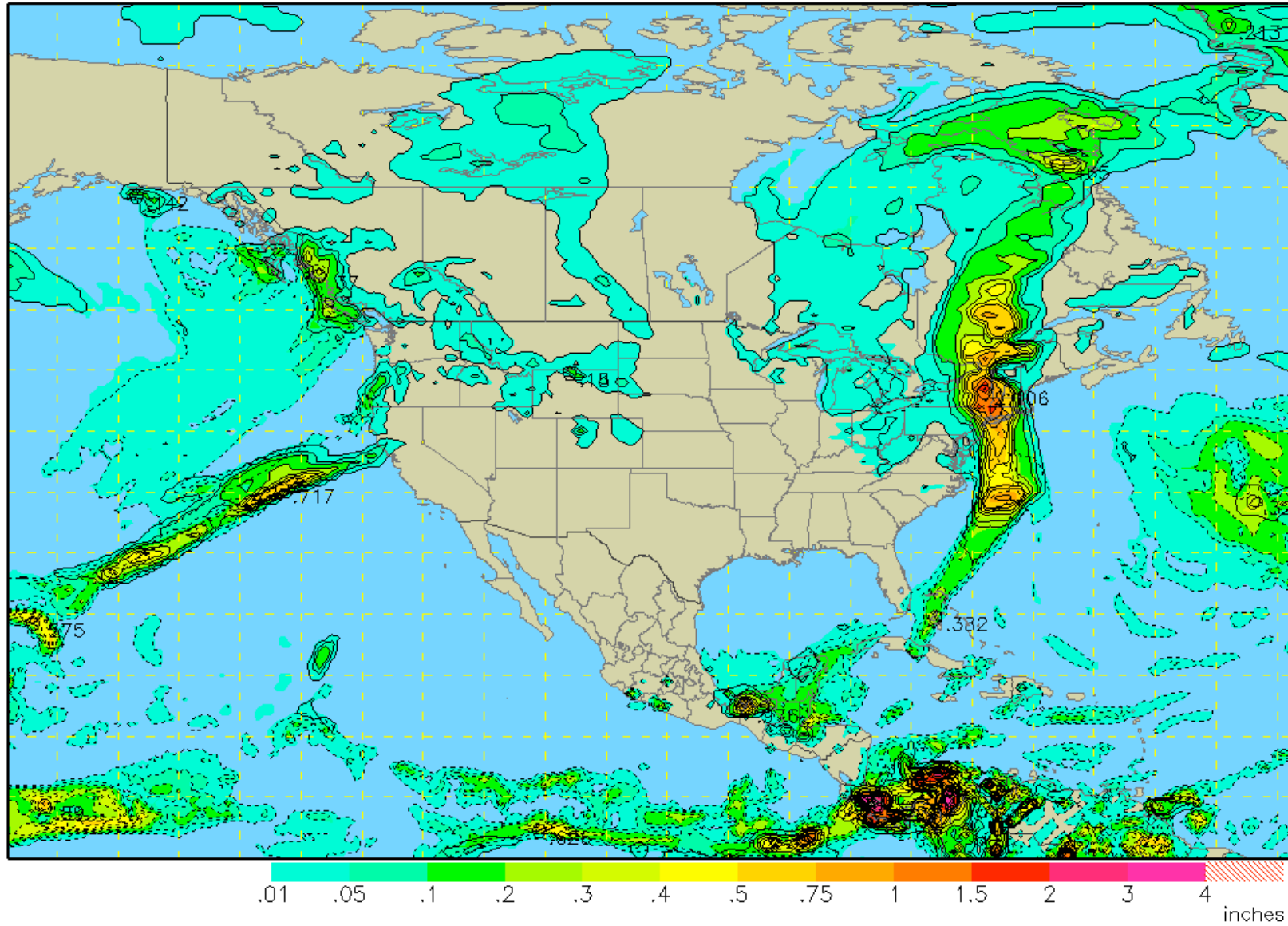
GFS (12z 28 Nov)



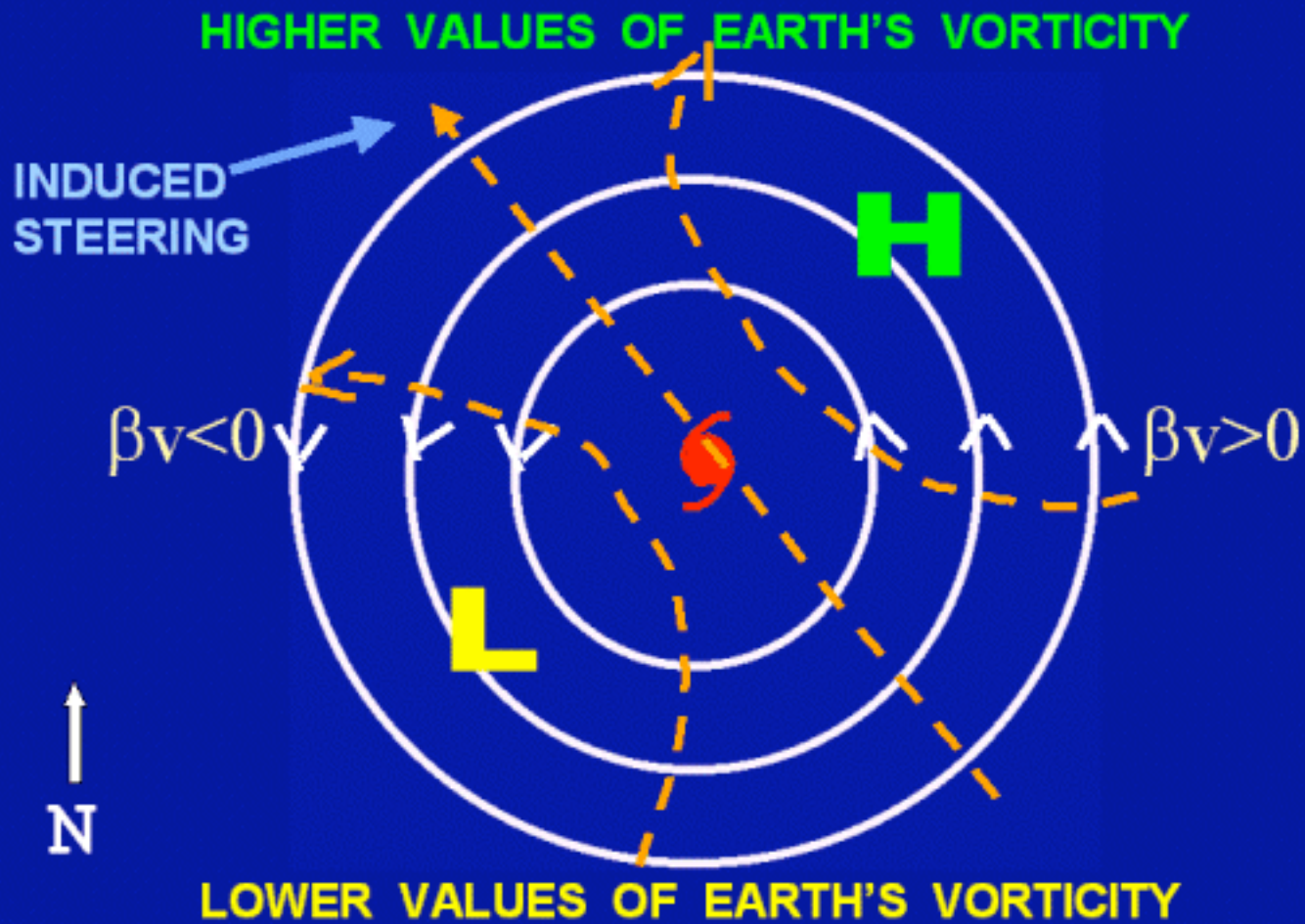
6-h accum precip (in) (total-shaded; convect-dashed)

84-hour forecast valid 0000 UTC Thu 02 Dec 2010

GFS (12z 28 Nov)



The Beta Effect

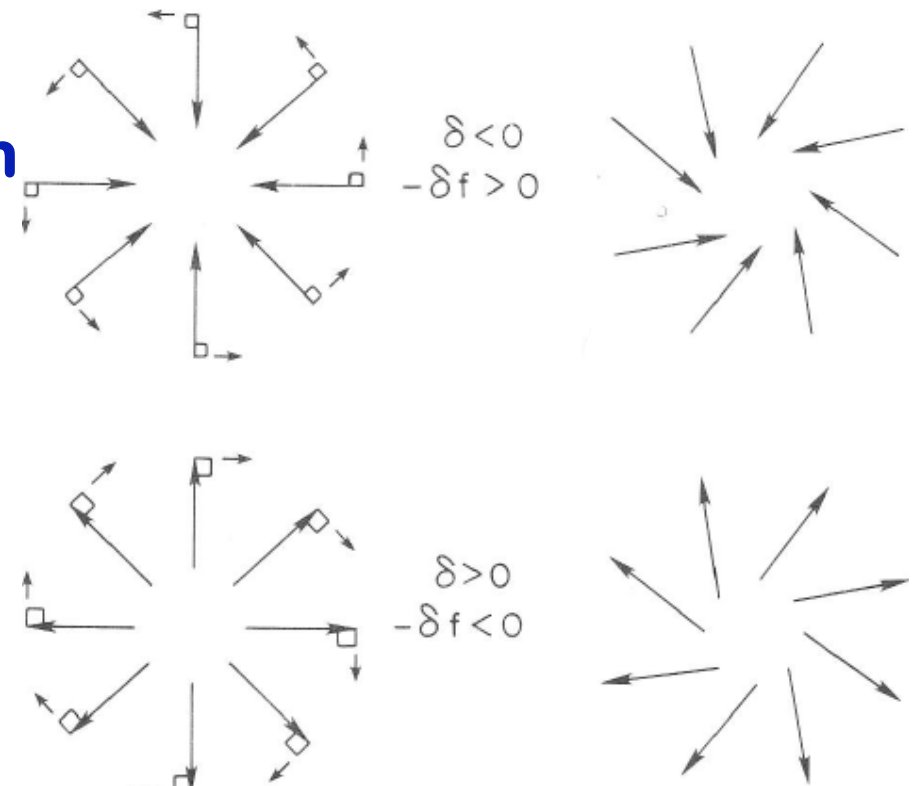


6) The divergence acting on planetary vorticity: $-f(\vec{\nabla} \cdot \vec{V})$

- Consider the purely convergent $(\vec{\nabla} \cdot \vec{V}) < 0$ and divergent $(\vec{\nabla} \cdot \vec{V}) > 0$ wind fields shown to the right.

- Because the Coriolis force deflects to the right of motion (small arrows in the figures) the winds are turned such that purely convergent flow attains a counterclockwise (cyclonic) spin, $\frac{\partial \vec{\xi}}{\partial t} > 0$ and purely divergent flow becomes divergent and clockwise (anticyclonic), $\frac{\partial \vec{\xi}}{\partial t} < 0$

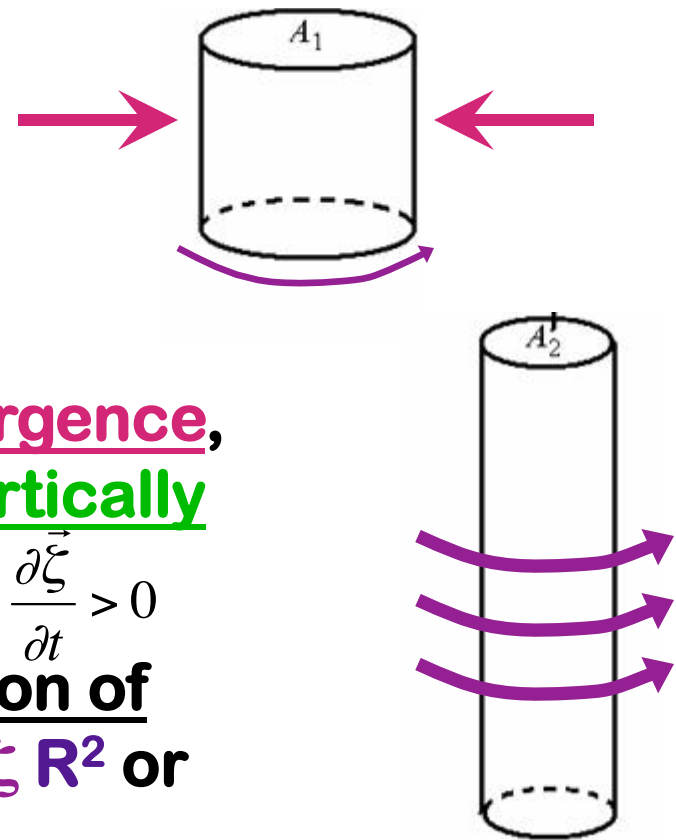
Convergence



Divergence

2) The divergence acting on relative vorticity: $-\xi(\vec{\nabla} \cdot \vec{V})$

- This term is often referred to as vortex tube stretching or shrinking and is analogous to the conservation of angular momentum (the ice skater again!).
- Consider the vertical columns of air (vortex tubes) to the right.
- The column on the left (top) is rotating with some vorticity $\xi > 0$.
- If the column is squished by convergence, $(\vec{\nabla} \cdot \vec{V}) < 0$ the column will stretch vertically and the rotation rate will increase $\frac{\partial \xi}{\partial t} > 0$
- This is equivalent to the conservation of angular momentum if we think of ξR^2 or ξA needing to be conserved.



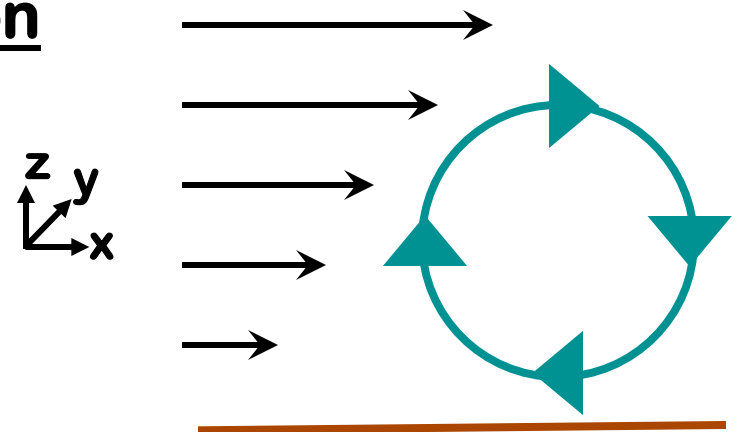
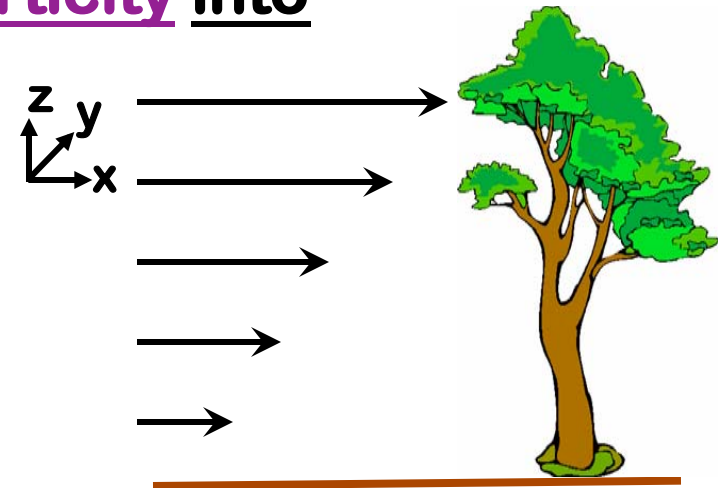
3) The tilting of horizontal shear vorticity: $-\hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z}$

- This term is also known as the twisting or tipping term and represents the production of vertical relative vorticity by literally tipping horizontal shear vorticity into the vertical.

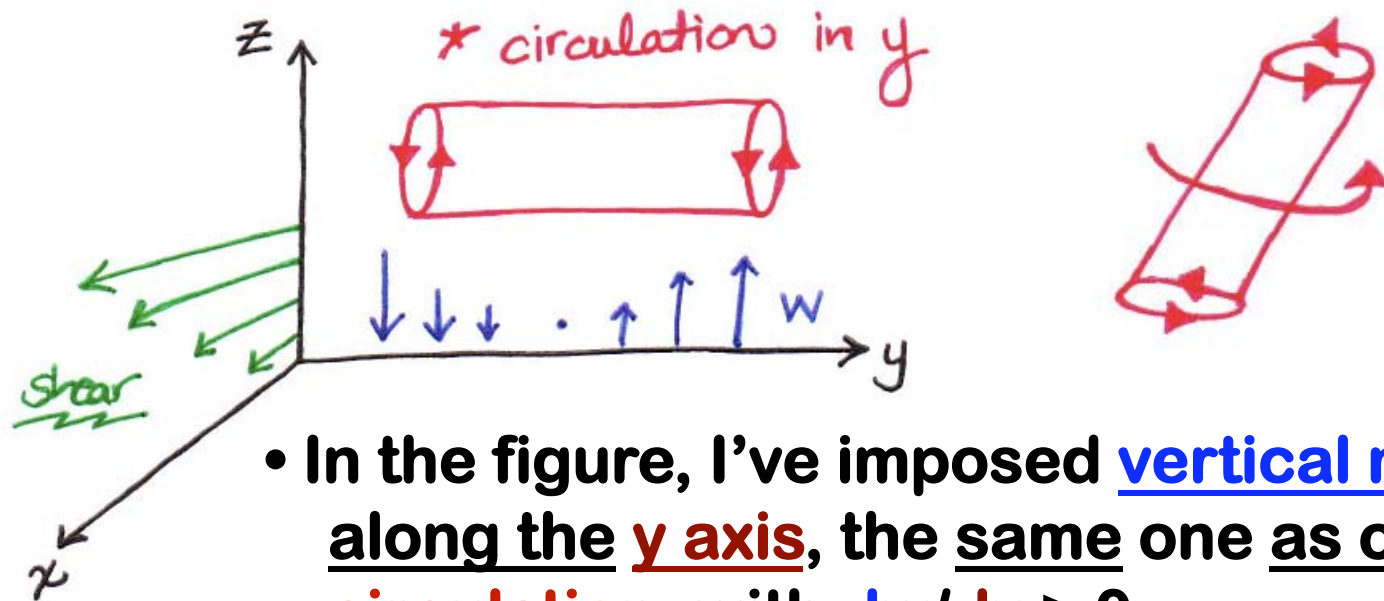
- Suppose, we have vertical speed shear of the wind in the x direction such that $\frac{du}{dz} > 0$, the westerly winds increase speed with height.

- From natural coordinates, we know speed shear normal to the direction of the flow generates vorticity.

- In this case, it is a clockwise circulation about the y axis, which is directed into the slide.



3) The tilting of horizontal shear vorticity: $-\hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z}$



- In the figure, I've imposed vertical motion along the y axis, the same one as our circulation, with $dw/dy > 0$.
- Thus, the horizontal circulation will be differentially tilted into the vertical, generating vertical vorticity. Here,

$$\frac{\partial \vec{\zeta}}{\partial t} = -\hat{k} \cdot \vec{\nabla}_w \times \frac{\partial \vec{V}}{\partial z} = -\hat{k} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{\partial w}{\partial y} & 0 \\ \frac{\partial u}{\partial z} & 0 & 0 \end{vmatrix} = -\left(-\frac{\partial u}{\partial z} \frac{\partial w}{\partial y} \right) > 0$$