1. The mean virtual temperature ($\bar{T}_v$) of the 1000-850 mb layer is 0°C. What is its thickness according to the hypsometric equation? ($R_d = 287$ J kg$^{-1}$ K$^{-1}$; $g_0 = 9.81$ m s$^{-2}$.)

**Ans:** Simple application of the hypsometric equation; answer is $\approx 1300$ gpm.

2. The thickness of a certain layer is 5000 gpm (geopotential meters). Layer $\bar{T}_v$ is 0°C. If the layer bottom pressure is 1000 mb, what is the pressure at the top of the layer?

**Ans:** Use the hypsometric equation to solve for $p_2$; the answer is about 535 mb.

3. In January, the troposphere is 18 km deep at the equator and 8 km deep at the pole. Tropopause pressures at the two locations are 100 and 300 mb, respectively. Calculate the mean $\bar{T}_v$ of the equatorial and arctic tropospheres, expressed in Kelvin, using the hypsometric equation and taking the surface pressure to be 1000 mb at both locations. In the arctic case, what difference is caused by presuming the surface pressure is 1030 mb instead?

**Ans:** More straightforward applications of the hypometric equation. Answers: tropical $\bar{T}_v = 267K = -6$°C; arctic $\bar{T}_v = 227K = -46$°C. Increasing the arctic surface pressure alters the mean layer temperature there only by a few degrees, to 221.5K. But think about this: why did the temperature have to be lower?

4. A barometer at the top of Mt. Rainier, at elevation 4.8 km above sea-level, reports a pressure of 550 mb. Station temperature is -10°C. Calculate the station’s sea-level pressure (SLP). The atmosphere is completely dry and you can (and should) presume the atmosphere is hydrostatic. You can also neglect trying to mass-weight the average temperature between station elevation and sea level, should assume the standard atmosphere’s tropospheric lapse rate of 6.5 K/km, and take $g_0 = 9.81$ m/s$^2$. Detail your assumptions and show your work.

**Ans:** Station temperature $T_s = 263K$. Assuming a lapse rate of 6.5 K/km over the station elevation of 4.8 km results in an estimated sea-level temperature of 294.2 K. This yields an arithmetic layer average temperature of $\bar{T} = 278.6$ K. As the atmosphere is dry, we are not worrying about virtual temperature. Therefore, recognizing the LHS is station elevation $Z$ in the hypsometric equation

$$Z = \frac{R_d \bar{T}}{g_0} \ln \left( \frac{p_{SLP}}{p_s} \right),$$

where $p_s$ is station pressure, we can use the exponential function to solve for $p_{SLP}$:

$$p_{SLP} = p_s \exp \left[ \frac{g_0 Z}{R_d T} \right].$$

Plugging in the numbers results in a sea level pressure of 991 mb.

5. I could solve the preceding problem using the altimeter equation,

$$p_{SLP} = p_s \left[ \frac{T_s}{T_s + \Gamma z_s} \right]^{(-\frac{g_0 T}{R_d T})},$$

where $p_s$, $z_s$, and $T_s$ are the station pressure, elevation, and temperature, respectively; $\Gamma$ is the tropospheric temperature lapse rate (taken to be 6.5°C/km); $g = g_0 = 9.81$ m/s$^2$; and $p_{SLP}$
is the sea-level pressure. Use this equation to verify your answer in the preceding problem. Then, derive this expression, starting with the hydrostatic equation, using the ideal gas law in the form \( p = \rho R_d T \) where \( R_d = 287 \) J kg\(^{-1}\) K\(^{-1}\), and assuming the temperature is linear between elevations \( z = z_s \) and sea-level \((z = 0)\). (That is, I’d take \( T = T_s + \Gamma (z_s - z)\).)

**Ans:** As the atmosphere is dry, we are not worrying about virtual temperature. We can start with the hydrostatic equation, after substituting for density using the ideal gas law

\[
\frac{dp}{p} = -\frac{g}{R_d T} dz.
\]

Substitute in \( T = T_s + \Gamma (z_s - z) \) and integrate between sea level \((z = 0, p = p_{SLP})\) and station elevation \((z = z_s\) and \(p = p_s)\):

\[
\int_{p_{SLP}}^{p_s} \frac{dp}{p} = -\frac{g}{R_d} \int_{z=0}^{z=z_s} \frac{dz}{T_s + \Gamma z_s - \Gamma z}.
\]

We can solve the RHS using this integral type

\[
\int \frac{c}{b + ax} dx = \frac{c}{a} \ln |ax + b| + C,
\]

where in our situation \( a = -\Gamma, b = T_s + \Gamma z_s, \) and \( c = 1 \). This brings us to

\[
\ln \frac{p_s}{p_{SLP}} = -\frac{g}{R_d} \left[ \frac{1}{\Gamma} \ln |T_s + \Gamma z_s - \Gamma z| \right]_{z=0}^{z=z_s}.
\]

After reversing the ratio on the LHS and evaluating the RHS, we obtain:

\[
-\ln \frac{p_{SLP}}{p_s} = \frac{g}{R_d \Gamma} \ln \left( \frac{T_s}{T_s + \Gamma z_s} \right).
\]

Use the log rule \( y \ln x = \ln x^y \), and then the exponential function to clear the logs, and finally solve for the sea level pressure. This results in

\[
p_{SLP} = p_s \left[ \frac{T_s}{T_s + \Gamma z_s} \right] \left( -\frac{g}{R_d \Gamma} \right)
\]

which concludes the proof after we appreciate that \( g = g_0 \).

6. In the standard troposphere, the temperature lapse rate is about 6.5° C/km and density decreases exponentially with height. Suppose Bizarro World is identical to Earth \((g = 9.81 \text{ m s}^{-2}, R_d = 287 \text{ J kg}^{-1} \text{ K}^{-1}, \) and hydrostatic) except density is constant with height in its atmosphere. What temperature lapse rate \((-\frac{dT}{dz})\) does this require? Hint: Start with the ideal gas law, and differentiate it with respect to height. Evaluate your lapse rate and express it in °C/km.

**Ans:** Density is constant, so if we differentiate the IGL with respect to height, we obtain

\[
\frac{dp}{dz} = \rho R_d \frac{dT}{dz}.
\]

Using the hydrostatic equation on the LHS gets rid of density, leaving

\[
\frac{dT}{dz} = \frac{-g}{R_d}.
\]

This evaluates to -0.034 K/m or -34 K/km. The lapse rate is \(-\frac{dT}{dz}\), incorporating the minus sign, so the lapse rate is 34°C/km. Enormous.