ATM 316 - Balanced flow

Fall, 2016 – Fovell

The geostrophic wind (V_g) is the combination of pressure gradient (PGF) and Coriolis forces that creates a straight-line wind, parallel to the isobars. In this context, if air is seen to curve, in order to follow curved isobars, then another force must be acting – such as the centripetal force. The addition of this third force will cause the wind speed to change, becoming either stronger than the geostrophic wind for the same pressure gradient (supergeostrophic) or weaker (subgeostrophic). In either case, the wind that represents the three force balance of PGF, Coriolis and centripetal is called the *gradient wind*, which we will designate as V.

How does flow curvature affect the wind speed? Intuition suggests that, if anything, counterclockwise (CCW) bending should accelerate the flow since we observe cyclones associated with larger wind speeds than anticyclones. However, the following analysis demonstrates that – for a given pressure gradient – **CCW curvature slows the wind** ($V < V_g$), rendering it subgeostrophic, while clockwise (CW) curvature permits a supergeostrophic speed. The reason why we see stronger winds around cyclones is because they invariably have larger pressure gradients.

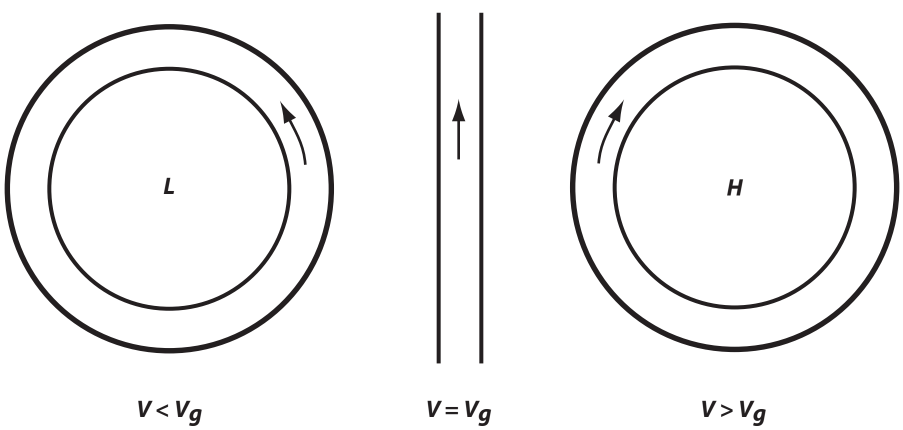

Figure 1: CCW, straight-line and CW curved flow parallel to isobars. V_g represents the geostrophic wind velocity; V is called the gradient wind.

The analysis is considerably simplified if we adopt a "natural coordinate system", one in which the coordinates follow the flow, so the horizontal velocity vector only ever has but a single component, pointing in the direction of the wind. Figure 2 illustrates a curving flow with coordinate axes $\hat{n}$ and $\hat{l}$ directed perpendicular and parallel to the flow, respectively. By convention, we take the $+\hat{n}$ direction to be to the left of the flow. In this framework, the vector velocity is $\vec{V} = V\hat{l}$, where V is a nonnegative speed. Where the flow is not straight-line, we can specify a finite radius of curvature, R. The convention is to take $R > 0$ ($R < 0$) when the flow is CCW (CW).

We start by recognizing that the acceleration following a parcel path ($\frac{d\vec{V}}{dt}$) is determined by PGF and Coriolis forces. In the natural coordinate system, the PGF potentially has components in the $\hat{n}$ and $\hat{l}$ directions. Here, we specify the flow as being parallel to isobars, so there is no pressure change in the $\hat{l}$ direction and PGF $= -\frac{1}{\rho}\frac{\partial p}{\partial n}\hat{n}$. Coriolis force acts to the right following the motion,

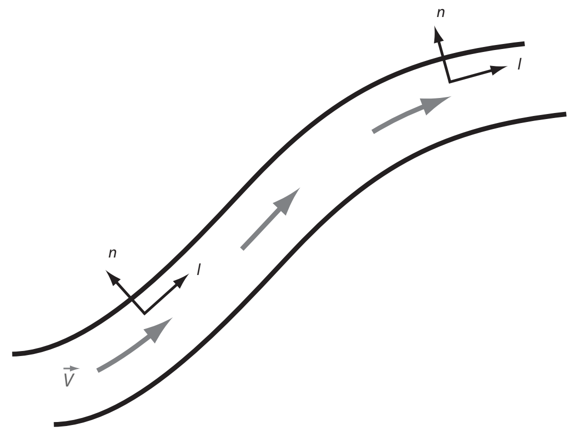

Figure 2: Natural coordinate system.

so it is pointing in the $-\hat{n}$ direction. Thus, it is $-fV\hat{n}$. Therefore,

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial n}\hat{n} - fV\hat{n}. \tag{1}$$

Next, we examine $\frac{d\vec{V}}{dt}$ and recognize that

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}(V\hat{l}) = \frac{dV}{dt}\hat{l} + V\frac{d\hat{l}}{dt}. \tag{2}$$

We have seen something like $\frac{d\hat{l}}{dt}$ before, when we dealt with centripetal acceleration. In that case, we looked at how a velocity vector shifted as it moved in a curved path; here, we're interested in how the unit vector $\hat{l}$ changes orientation. Paralleling that derivation, we find

$$\frac{d\hat{l}}{dt} = \left|\hat{l}\right|\frac{d\theta}{dt}\hat{n}.$$

Here, the magnitude of the unit vector is one, $\frac{d\theta}{dt} = \omega$ and $\omega = \frac{V}{R}$, so

$$\frac{d\hat{l}}{dt} = \frac{V}{R}\hat{n}.$$

After inserting this expression into (2), combining the result with (1) and collecting the $\hat{n}$ terms, we have

$$\frac{V^2}{R} + fV = -\frac{1}{\rho}\frac{\partial p}{\partial n}. \tag{3}$$

We can always define a wind that is in precise geostrophic balance with the pressure gradient. So, calling this wind V_g, we have $fV_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial n}$, which can be substituted into (3), resulting in

$$\frac{V^2}{R} + fV - fV_g = 0. \tag{4}$$

To see why CCW curvature mathematically results in a subgeostrophic wind, divide (4) by V and rearrange the result as follows:

$$\frac{V_g}{V} = 1 + \frac{V}{fR}.$$

Recall V is a nonnegative speed and $R > 0$ for CCW motion. Thus, $V_g > V$ for cyclonic flow since the $\frac{V}{fR}$ term is positive in the Northern Hemisphere. Similarly, for CW motion, $R < 0$ and $V > V_g$. For the same pressure gradient, anticyclonic flow is supergeostrophic.

That was the math, now let's look at the physics. Consider the northbound parcel illustrated in the figure below. If the isobars were not curved, the parcel would move straight, as indicated by the dashed line. That's the parcel's inertial motion in this reference frame, and its speed would be geostrophic. But note that given the arrangement of isobars, this path would actually carry the parcel *away* from lower pressure, and towards higher pressure. As the PGF is always directed towards low pressure, the component of this force starts acting AGAINST the flow motion as the parcel starts crossing isobars. *That slows the parcel down*, so $V < V_g$. The Coriolis force is reduced, since that is proportional to parcel speed. Thus, these two now unbalanced forces create an acceleration, directed inward towards the center of spin, that we can identify as the centripetal force. The CCW curving gradient wind is subgeostrophic for the same PGF; we have just seen physically why the flow had to slow down, and the consequences of that speed change is direction change.

The same reasoning leads to why a parcel heading CW around high pressure is supergeostrophic (not shown). Inertia would carry the parcel towards lower pressure, causing it to speed up ($V > V_g$) as a component of the PGF starts acting in the same direction as the motion. As the flow speeds up, the Coriolis force is increased. The imbalance in this case bends the parcel to the right, away from its straight-line path and into a CW curving orbit around high pressure at supergeostrophic speeds.

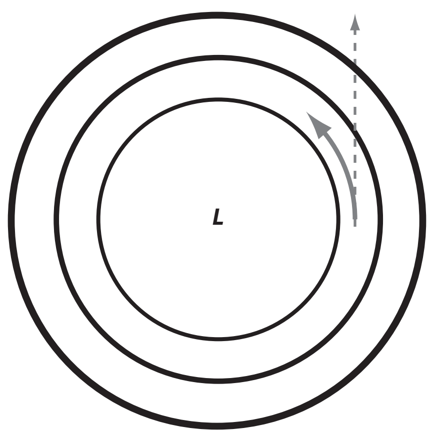

Figure 3: Path of a parcel moving amid curved isobars, versus the path representing its inertia.

Two other interesting results obtain from (3) and (4). If the horizontal pressure gradient vanishes, the remaining terms represent *inertial flow*. This flow is always clockwise in the Northern Hemisphere since (3) implies that

$$R = -\frac{V}{f}.$$

Since V is nonnegative and $f > 0$ in the Northern Hemisphere, $R < 0$, indicating CW motion. A parcel moving at speed V in a circular path of radius R has a period P of

$$P = \frac{2\pi R}{V} = \frac{2\pi}{f}.$$

At 45°N, the period is about 17 h; closer to the equator, this so-called inertial oscillation period is several days in length. This has relevance to ocean circulations, where pressure gradients are very often far, far smaller than in the atmosphere.

In contrast, *cyclostrophic flow* occurs when the scale of circulation is sufficiently small (in time and space) to ignore the Coriolis force. In this situation, (3) becomes

$$\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n},$$

from which it follows that

$$V = \sqrt{-\frac{R}{\rho}\frac{\partial p}{\partial n}}.$$

This equation represents both CCW and CW motion around low pressure. If pressure decreases in the $\hat{n}$ direction, then $R > 0$ keeps V real, and that's CCW flow around low pressure. If pressure increases with $\hat{n}$, then R has to be negative, which means CW flow around low pressure. This is the balance of forces relevant to a tornado or any other small scale circulation, including vortices you can create in a teacup. It doesn't matter which way you stir your tea, you create low pressure at the circulation center.