Terminal Velocity of Raindrops Aloft[1]

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ABSTRACT

The inadequacy of previous calculations of terminal velocities at other than sea level conditions is discussed. Attention is called to actual measurements of terminal velocities at different air densities, and empirical formulae are presented which fit the data very closely.

1. Introduction

Accurate measurements of the terminal velocity of water drops at 20C and 1013 mb were made by Gunn and Kinzer (1949), and their measurements are used almost universally by meteorologists today. For many applications it is desirable to have reliable estimates at other than sea level conditions. The larger terminal velocity of raindrops aloft is important in many cloud-physics computations, and has a critical role in the interpretation of Doppler radar records.

Several investigators have published estimates of terminal velocities aloft, based on assumptions which will here be shown to be inadequate. The present paper is meant to draw attention to what appear to be the only actual measurements of terminal velocities at low air densities, those of Davies,[3] and to present convenient empirical formulae which fit Davies' data very closely.

2. Calculation of terminal velocity from drag data

For a water drop falling at its terminal velocity in air, the equation of motion reduces to

$$mg=\tfrac{1}{2}\rho V^2 C_D \pi r^2, \tag{1}$$

which states that the drop's weight is balanced by the upward-directed aerodynamic drag. Here m is the mass of the drop, g the acceleration of gravity, ρ air density, V the terminal velocity of the drop, C_D the drag coefficient, and r the drop radius. For nonspherical liquid drops the radius of a sphere of equal volume is normally used, and C_D is adjusted accordingly. Clearly, the problem of using (1) to predict V is centered around the determination of C_D, which for a rigid body, is a function only of the Reynolds number Re. The usual method of calculating terminal velocities, then, is equivalent to the following indirect procedure. One can assume a value for V and calculate the Reynolds number from

$$\mathrm{Re}=\frac{2rV}{\nu}, \tag{2}$$

where ν is the kinematic viscosity. From the appropriate drag curve, such as given in Fig. 1, one obtains C_D, and a new V is calculated from (1). Using this velocity, the calculation can then be repeated. With a reasonable first guess the method will converge with only a few iterations. While this iterative procedure is eliminated in a scheme proposed by McDonald (1960a), the underlying physics is identical to that of the procedure just outlined; the foregoing also appears to be the method used by Mason (1957), Caton (1966) and du Toit (1967). Battan (1964) used the ratio of terminal speeds of spheres aloft and at surface conditions to adjust the Gunn and Kinzer data to other air densities. However, Battan used kinematic viscosities applicable in the standard atmosphere, rather than at the temperatures and pressures he specified.

It is important now to see why the procedure outlined above will be inadequate for computing velocities of drops greater than about 1 mm in diameter, above which the drop shape starts to deviate from sphericity. Consider the drag curve determined by Gunn and Kinzer for water drops, shown in Fig. 1. For $\mathrm{Re}<500$, the smaller drag coefficients of water drops than of rigid spheres is evidence of an organized circulation within the drop (McDonald, 1960b; Kintner, 1963). For $\mathrm{Re}\gtrsim 500$, water drops have a higher C_D than do rigid spheres. This is caused by the well-known flattening of large drops with an increase in horizontal cross section. The effect is enhanced by the customary use of the equivalent spherical radius in Eq. (1).

Thus, as increasingly larger drops take on increasingly flatter shapes, the drag curve becomes quite steep, in contrast to that of a rigid sphere. For large

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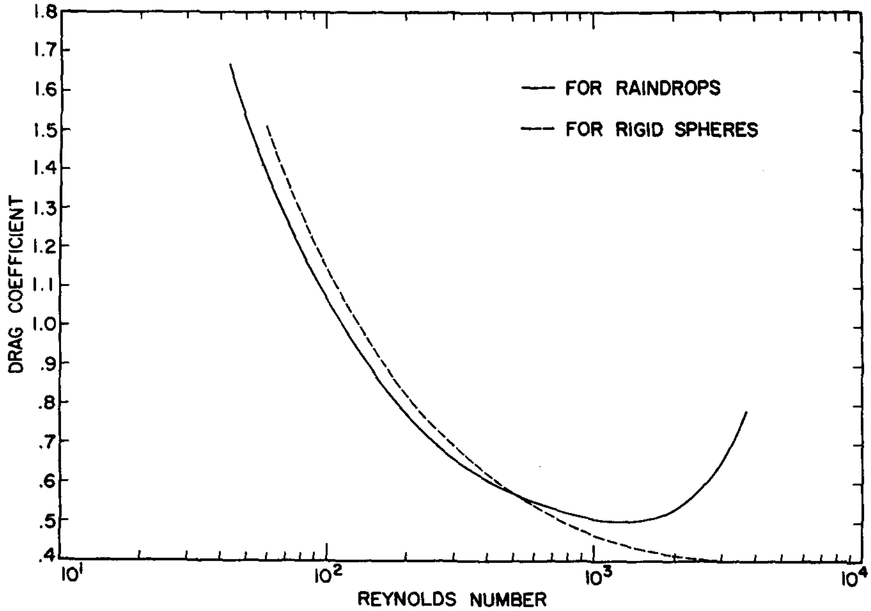

FIG. 1. Drag curves for raindrops and for rigid spheres. The solid curve was computed from the Gunn and Kinzer measurements.

drops, the drag coefficient is not a function of the Reynolds number alone, as for rigid bodies, but also depends on other nondimensional numbers associated with the drop-shape problem.

We are now in a position to see why the method used by earlier investigators is in error. Essentially any reasonable assumption about C_D will predict a larger terminal velocity for a drop falling in less-dense air, but corresponding to this, actually a smaller Reynolds number. In the size range of large raindrops, this smaller Re will dictate use of a smaller C_D, a drag coefficient which was actually determined at sea level for a smaller and, hence, less flattened drop. This C_D will be too small, and the method will predict excessively large terminal velocities. Thus, the Gunn and Kinzer drag curve cannot be used to extrapolate terminal velocity data for drops $\gtrsim$1mm in diameter to other atmospheric conditions.

The difficulty with theoretically extending terminal velocities to lower air densities is concerned with knowing the equilibrium drop shape at these new conditions. If the drop shape were known, one could obtain drag coefficients from suitable wind tunnel measurements. However, while the physics of raindrop shape is moderately well understood (McDonald, 1954; Foote, 1969), there is great difficulty in actually predicting equilibrium drop shapes because of the difficulty in determining the aerodynamic flow around the (equilibrium) drop surface, which governs the external pressure distribution over the drop. Even despite the complications arising due to the interrelation of drop shape and external flow, which would make determination of the true equilibrium very difficult, there are currently no numerical methods which will treat high Reynolds number flows around bodies of arbitrary shape (for which no special coordinate systems exist).

Thus, at the present time, actual measurement of terminal velocities is the only feasible approach.

3. Davies' data for terminal velocities aloft

Apparently the only actual measurements of terminal velocities at other than sea-level densities are those of Davies (*loc. cit.*), reported by Sutton (1942), and also published by Best (1950). Briefly, Davies made use of a fall tube 11 m in length, which could be evacuated down to about half an atmosphere. Velocities were measured by allowing the drops to fall through six narrow sharp-edged beams of light. From the time of passage through successive beams an average velocity could be computed, and a check could be maintained on the absence of acceleration.

In a somewhat neglected paper, Best (1950) tabulated all of Davies' experimental data, and gave empirical formulae which were fitted to Davies' curves. For terminal velocities at surface conditions, Best proposed use of the equation,

$$V_0(d)=A_0\{1-\exp[-(d/a)^n]\}, \tag{3}$$

where V_0 refers to surface conditions, and A_0, a and n are constants that take the values 943, 1.77 and 1.147, respectively, when V_0 is in cm sec^{-1} and the drop diameter d is in mm. As an extension to other altitudes, Best proposed

$$V(d)=V_0(d)e^{bz}, \tag{4}$$

where V_0 comes from (3), and b is a constant which depends on atmospheric conditions. Best gave values of b of 0.0405 for the I.C.A.N. standard atmosphere and 0.0354 for the standard summer tropical atmosphere.

Sutton (1942) quotes the following expression, due to Davies (*loc. cit.*), which is supposed to predict the terminal velocities of drops of any liquid down to pressures of half an atmosphere (see also Mason, 1957):

$$\log_{10} \mathrm{Re} = 2.655\left[\frac{\log_{10} Q - f_1}{f_2}\right]^{\frac{1}{2}} - f_3, \tag{5}$$

where

$$f_1 = 0.460 + 1.012 \log_{10} X + 0.225 (\log_{10} X)^2,$$
$$f_2 = 0.933 - 0.167 \log_{10} X,$$
$$f_3 = 3.306 + 0.118 \log_{10} X + 0.0765 (\log_{10} X)^2,$$
$$X = \rho_l d^2 g/\sigma,$$
$$Q \equiv C_D \mathrm{Re}^2 = 8mg/(\pi\rho\nu^2).$$

Here ρ_l is the density of the liquid, and all other symbols are as previously defined. This relation is stated to be accurate to within 3% for all liquids in the range

$$0.4 < \rho_l d^2 g/\sigma < 1.4. \tag{6}$$

For water drops, the limits on d in (6) become 1.8–3.3 mm. Calculations using (5), however, show that at typical 500-mb conditions it is in error by 4% for 1-mm drops, increasing to 17% for 5.8-mm drops, when compared with Davies' experimental data. For 700-mb conditions, it is within a few per cent for drops $\lesssim 1$ mm, but increases to a maximum error of about 14%. Eq. (5) overestimates velocities of water drops in all cases. Clearly, (5) is of little use for meteorological purposes.

While (4) is useful for analytical manipulation, it is written in terms of height, rather than the more relevant variable, air density (temperature is also a parameter of secondary importance because of viscosity dependence on temperature). In an attempt to arrive at an expression that will predict terminal speeds at arbitrary atmospheric conditions (rather than at a given height in the standard atmosphere), the authors have gone back to the original Davies' data to derive the more general equation

$$V(d) = V_0(d) 10^Y \left[1 + 0.0023\left(1.1 - \frac{\rho}{\rho_0}\right)(T_0 - T_0)\right], \tag{7}$$

where

$$Y = 0.43 \log_{10}\left(\frac{\rho_0}{\rho}\right) - 0.4\left[\log_{10}\left(\frac{\rho_0}{\rho}\right)\right]^{2.5}. \tag{8}$$

The subscript zero refers to 20C and 1013 mb. Several curves computed from (7) are shown in Fig. 2. The Gunn and Kinzer values have been used for V_0.

The bracketed term in (7) attempts to take account of the dynamic viscosity dependence on temperature, which changes, by a small amount, the Reynolds number, and, hence, the drag coefficient. All Davies' data were taken at 20C.

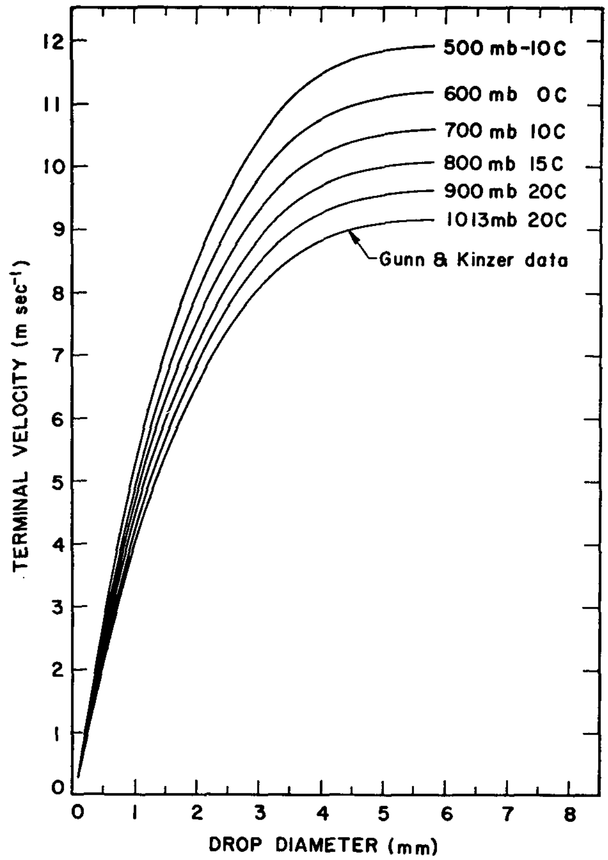

FIG. 2. Terminal velocities computed from Eq. (7) for indicated conditions. The Gunn and Kinzer curve is also shown.

The factor in (7) involving Y is the important variable. The form exhibited there is simply an improvement on the simpler, but slightly less accurate form,

$$V(d) = V_0(d)\left[\frac{\rho_0}{\rho}\right]^{0.4}. \tag{9}$$

If a given drop had the same drag coefficient at different air densities, (1) shows that the exponent in (9) would be 0.5. The smaller exponent of (9) implies that drops falling at successively lower air densities must have correspondingly higher drag coefficients. In fact, the drag coefficients calculated from Davies' data, seen in Fig. 3, show this very clearly, and illustrate the nature of the error made in using Gunn and Kinzer surface data to extrapolate terminal speeds aloft. The shape of these curves in Fig. 3 is evidence for the fact that raindrops are actually flatter high in the atmosphere than at sea level.

The viscosity correction term in (7) was found by computing terminal velocities from (1) using the drag curves in Fig. 3. This correction varies somewhat with

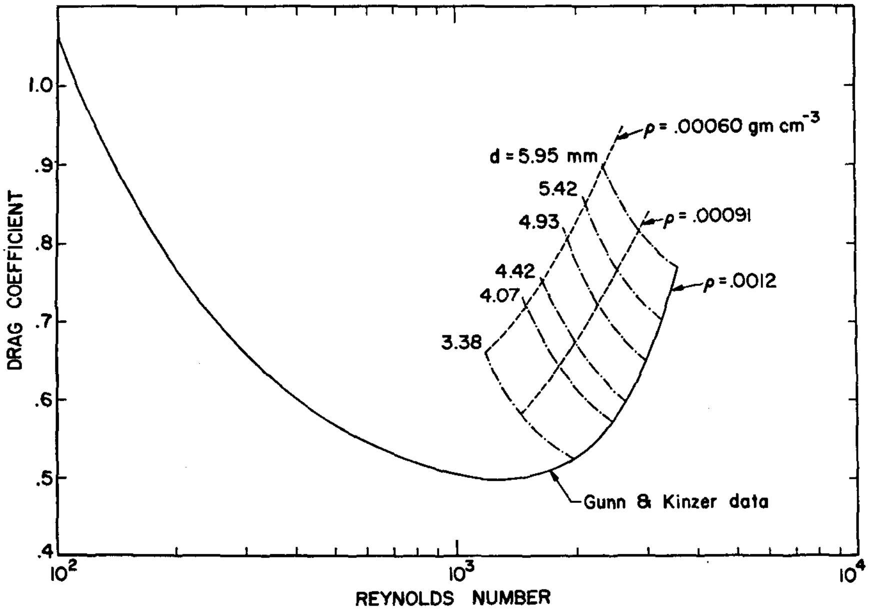

FIG. 3. Drag data computed from Davies' measurements. The dashed lines show the drag curves at given air density. The dot-dash lines show the change in C_D with air density for drops of a given size. The Gunn and Kinzer curve from Fig. 1 is also shown.

altitude, and led to the factor there involving ρ. At typical 500-mb conditions in mid-latitudes, the term amounts to about a 3% correction. As a rule of thumb, at constant pressure a 10C change in temperature will make a 1% (or less) change in V.

The predictions of Eq. (7) using the Gunn-Kinzer data for V_0 are always within 2.5% of Davies' data over the size range of his observations (3.38–5.95 mm diameter). This maximum difference occurs at the lowest density used by Davies ($\rho=0.6\times10^{-3}$ gm cm^{-3}). Most of the discrepancy is explained by the 2% difference between the Gunn and Kinzer and the Davies surface data for certain drop sizes. The simplification of averaging V/V_0 over all diameters in (7) introduces no more than a 1% error.

No attempt was made to take account of the temperature dependence of σ, the surface tension coefficient.

4. Empirical equation for V_0

It is frequently convenient to have an empirical expression for terminal velocities at surface conditions. While Best's expression (3) gives values within 1% of Davies' data for drop diameters from 2.3 to 6.0 mm, it is in error by as much as 8% for smaller sizes. For these reasons it seems desirable to base an expression

TABLE 1. Coefficients in Eqs. (10) and (11), for d in mm and V in m sec^{-1} (the numbers given must be multiplied by 10^x where x is given in parentheses). The maximum difference between the polynomial prediction and the Gunn and Kinzer (1949) values is also given as the maximum error for each polynomial.

j	a_j $N=3$	a_j $N=5$	a_j $N=9$	b_j $N=13$
0	−1.9274(−1)	−3.1682(−1)	−8.5731540(−2)	1.6186834(−1)
1	4.9625(0)	5.4506(0)	3.3265862(0)	−4.5352824(−1)
2	−9.0441(−1)	−1.3806(0)	4.3843578(0)	1.0314424(0)
3	5.6584(−2)	2.3612(−1)	−6.8813414(0)	−5.0842472(−1)
4		−2.8781(−2)	4.7570205(0)	−7.3450430(−2)
5		1.6486(−3)	−1.9046601(0)	1.5748428(−1)
6			4.6339978(−1)	−5.8210528(−2)
7			−6.7607898(−2)	9.0774564(−3)
8			5.4455480(−3)	−5.8192169(−4)
9			−1.8631087(−4)	8.2413985(−5)
10				−3.2915963(−5)
11				5.4687449(−6)
12				−3.9849021(−7)
13				1.0944173(−8)
Maximum error	0.11 m sec^{-1}	0.07 m sec^{-1}	0.03 m sec^{-1}	0.1 mm

for V_0 on the slightly more accurate Gunn and Kinzer data. The present authors have found that an Nth degree polynomial of the form

$$V_0 = \sum_{j=0}^{N} a_j d^j, \tag{10}$$

will fit the Gunn and Kinzer data to any desired accuracy by simply increasing N. The a_j's are determined by using a least-squares curve fitting technique (see, e.g., McCracken and Dorn, 1964).

The values of the a_j's are given in Table 1 for three different polynomials, and the maximum error involved in using (10) in the size range 0.1–5.8 mm diameter is also given. As is seen from the signs of the a_j's, there is a good deal of compensation between the terms in (10). Indeed, for the higher order polynomials, the numerical work becomes prohibitive for anything but machine computation. For example, for $N=9$, a minimum of 7 digits need to be retained for accuracy in the second decimal place [individual terms in (10) become as large as 10^4 m sec^{-1}, summing to order 10 m sec^{-1}].

In Doppler radar studies it is frequently necessary to predict drop size given terminal velocity. If the calculations are to be carried out on a computer, an empirical expression is needed. The following has proved convenient:

$$d = \sum_{j=0}^{13} b_j V_0^j, \tag{11}$$

where the b_j's are also given in Table 1 (here V_0 must be in m sec^{-1} and d in mm). Eq. (11) was obtained from a least-squares fit to the Gunn and Kinzer data, and applies over the range 0.27–9.17 m sec^{-1}. It appears that nothing less than a thirteenth degree polynomial will give sufficient accuracy, i.e., predict d to within 0.1 mm. At least 7 digits must be carried in (11) for accuracy at all sizes.

For deduction of drop size from terminal velocities aloft, (7) can be used to find V_0, and this value then used in (11) to yield drop diameter.

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