The QG Thermodynamic Eqn

- last class, we derived the QG vorticity eqn:

$$\nabla_p^2\left[\frac{\partial \Phi}{\partial t}\right] = -f_0 \vec{V_g}\cdot\nabla\left[\frac{1}{f_0}\nabla_p^2\Phi + f\right] + f_0^2\frac{\partial \omega}{\partial p} + f_0 \hat{k}\cdot\vec{\nabla}\times F$$

(remember from your homework that $\zeta_g = \frac{1}{f_0}\nabla_p^2\Phi$)

- here, we will derive the QG <u>thermodynamic eqn</u>
- the goal will be to combine the QG thermo. and vorticity eqns to get the QG <u>height tendency eqn</u>

- start with the thermodynamic eqn (a statement of the conservation of energy)

$$\rightarrow \quad dQ = du + dw$$

<u>dQ: heat transfered</u>
- <u>$dQ>0 \rightarrow$</u> heat in
- <u>$dQ<0$</u> $\rightarrow$ heat out

<u>du: internal energy change</u>

<u>dw: work done</u>
- <u>$dw>0$</u> $\rightarrow$ work done by system
- <u>$dw<0$</u> $\rightarrow$ work done on system

- from thermo class:

$du = C_v dT$ (c_v: specific heat of air at constant volume)

$dw = p\,d\alpha$ (α: specific volume, $1/\rho$)

- sub in to get $dQ = C_v dT + p\,d\alpha$
  - substitute for $p\,d\alpha$:
    - ideal gas law: $p = \rho RT$

$$\rightarrow \frac{p}{\rho} = RT$$

$$\rightarrow p\alpha = RT$$

- take derivative of $p\alpha = RT$
  $\rightarrow$ get $p\,d\alpha = R\,dT - \alpha\,dp$

$$\rightarrow dQ = C_v dT + R\,dT - \alpha\,dp \qquad \boxed{C_p = C_v + R}$$

$$= (C_v + R)dT - \alpha\,dp \rightarrow C_p dT - \alpha\,dp$$

- divide by dT

$$\rightarrow \frac{dQ}{dt} = C_p \frac{dT}{dt} - \alpha \underbrace{\frac{dp}{dt}}_{\omega} = C_p \frac{dT}{dt} - \alpha\omega$$

- assuming hydrostatic and quasigeostrophic (i.e., $\vec{V}_{ag} << \vec{V}_g$) conditions, expand $\frac{dT}{dt}$

$$\rightarrow \frac{dQ}{dt} = C_p \overbrace{\left(\frac{\partial T}{\partial t} + \vec{V}_g \cdot \nabla_p T + \omega \frac{\partial T}{\partial p}\right)}^{dT/dt} - \alpha\omega$$

- divide by C_p

$$\rightarrow \frac{1}{C_p}\frac{dQ}{dt} = \frac{\partial T}{\partial t} + \vec{V}_g \cdot \nabla_p T + \omega \frac{\partial T}{\partial p} - \frac{\alpha}{C_p}\omega$$

$$\rightarrow \frac{1}{c_p}\frac{dQ}{dt} = \frac{\partial T}{\partial t} + \vec{V}_g \cdot \nabla_p T + \omega\left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\right)$$

· solve for $\frac{\partial T}{\partial t}$ (local temperature change):

$$\frac{\partial T}{\partial t} = -\vec{V}_g \cdot \nabla_p T - \omega\left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\right) + \frac{1}{c_p}\frac{dQ}{dt}$$

$$\rightarrow \boxed{\frac{\partial T}{dt} = -\vec{V}_g \cdot \nabla_p T + \omega\left(\underbrace{\frac{\alpha}{c_p}}_{\substack{\text{dry adiabatic} \\ \text{lapse rate,} \\ \Gamma_d}} - \underbrace{\frac{\partial T}{\partial p}}_{\substack{\text{environmental} \\ \text{lapse rate,} \\ \Gamma_{env}}}\right) + \frac{1}{c_p}\frac{dQ}{dt}}$$

Q6 Thermodynamic Eqn

· let's look at 2nd term on RHS → adiabatic cooling / warming term

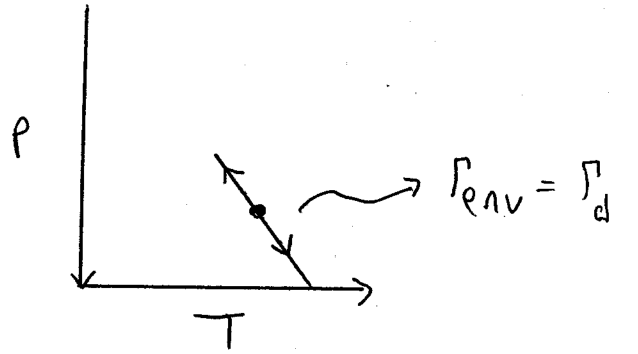
$$\frac{\partial T}{\partial t} \propto \omega\left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right)$$

· what if $\frac{\alpha}{c_p} = \frac{\partial T}{c_p}$? $\quad \Gamma_d = \Gamma_{env}$

→ environmental lapse rate is the dry adiabatic lapse rate (environment is well mixed)

$$\rightarrow \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = 0 \rightarrow \underline{\text{term is zero}}!$$

- in other words, no amount of vertical motion can cause a local temperature change



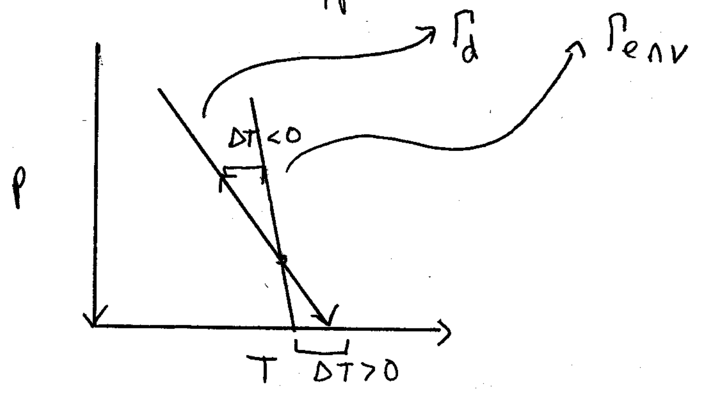
- upward motion results in cooling at dry adiabatic lapse rate
- sinking motion results in warming at dry adiabatic lapse rate

→ no change to environmental temperature profile

- what if $\frac{\partial T}{\partial p} < \frac{\alpha}{c_p}$? ($\Gamma_{env}$ slightly less than Γ_d)

$$\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \text{small positive}$$

→ small vertical motion yields small change in local temperature



$$\frac{\partial T}{\partial t} \propto \omega \underbrace{\left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right)}_{\text{small }+}$$

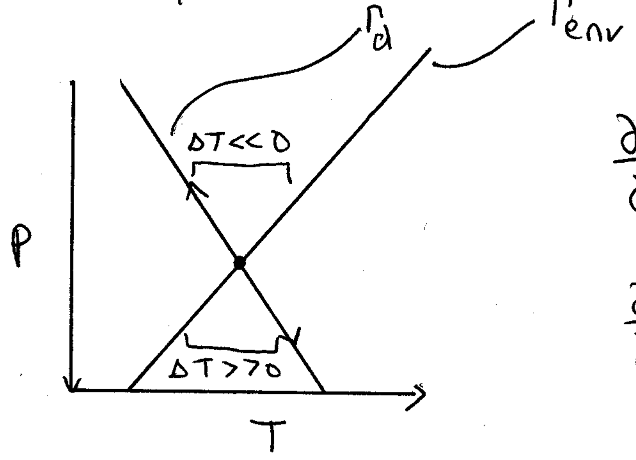
$$\frac{\partial T}{\partial t} \propto +\omega$$

$\omega < 0$, $\frac{\partial T}{\partial t} < 0$ rising motion: cooling

$\omega > 0$, $\frac{\partial T}{\partial t} > 0$ sinking motion: warming what if $\frac{\partial T}{\partial p} << \frac{\alpha}{c_p}$? (Γ_{env} much less than Γ_d)

$$\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \text{big positive}$$

→ small vertical motion yields big change in local temperature



large +

$$\frac{\partial T}{\partial t} \propto \omega \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right)$$

$$\frac{\partial T}{\partial t} \propto + \omega$$

$\omega < 0$, $\frac{\partial T}{\partial t} << 0$

$\omega > 0$, $\frac{\partial T}{\partial t} >> 0$

- note that $\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}$ (or $\Gamma_d - \Gamma_{env}$) is a modified version of the static stability parameter, σ

- using $T = \theta \left(\frac{p}{p_0} \right)^{R/c_p}$, $\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \rightarrow -T \frac{\partial \ln \theta}{\partial p}$

- static stability parameter, σ, defined as

$$\sigma = -\frac{R}{p} \left(\frac{T \partial \ln \theta}{\partial p} \right)$$

→ can now rewrite Q6 thermo. eqn with σ:

$$\frac{\partial T}{\partial t} = \underbrace{-\vec{V}_g \cdot \nabla_p T}_{\text{(A) geostrophic temperature advection}} + \underbrace{\omega \sigma \frac{p}{R}}_{\text{(B) adiabatic cooling / warming}} + \underbrace{\frac{1}{C_p}\frac{dQ}{dt}}_{\text{(C) diabatic cooling / warming}}$$

$\sigma = 0$ → neutral static stability
- vertical motion produces no temperature change

$\sigma > 0$ → low static stability
- small vertical motion produces small temp. change

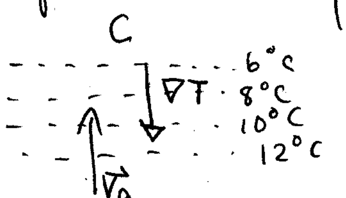
$\sigma >> 0$ → high static stability
- small vertical motion produces large temp. change

A: $\frac{\partial T}{\partial t} \propto -\vec{V}_g \cdot \nabla_p T$

→ like geostrophic vort. advection term in Q6 vorticity eqn

- V_g & $\nabla_p T$ point in same direction, $\frac{\partial T}{\partial t} < 0$

- V_g & $\nabla_p T$ point in opposite direction, $\frac{\partial T}{\partial t} > 0$



→ warm air advection

B: $\frac{\partial T}{\partial t} \propto \omega \sigma \frac{p}{R}$

- see earlier notes

c: $\frac{\partial T}{\partial t} \propto \frac{1}{c_p} \frac{dQ}{dt}$

$\frac{dQ}{dt} > 0$ (diabatic heating) $\rightarrow \frac{\partial T}{\partial t} > 0$

$\frac{dQ}{dt} < 0$ (diabatic cooling) $\rightarrow \frac{\partial T}{\partial t} < 0$

<u>diabatic heating</u>:
- condensation
- heating from earth's surface

<u>diabatic cooling</u>:
- evaporation
- radiative cooling (could be from cloud tops or at surface)

- can write Q6 thermodynamic eqn in terms of thickness instead of temperature

$\rightarrow$ hypsometric eqn

$$\rightarrow T = \frac{p}{R} \underbrace{\left(-\frac{\partial \Phi}{\partial p}\right)}_{\substack{\text{thickness} \\ \text{(press. coordinates)}}}$$

- sub in $\frac{p}{R}\left(-\frac{\partial \Phi}{\partial p}\right)$ for T

$$\rightarrow \frac{\partial}{\partial t}\left(-\frac{p}{R}\frac{\partial \Phi}{\partial p}\right) = -\vec{V}_g \cdot \nabla\left(-\frac{p}{R}\frac{\partial \Phi}{\partial p}\right) + \omega\sigma\frac{p}{R} + \frac{1}{c_p}\frac{dQ}{dt}$$

- multiply by $\frac{R}{p}$
- note that $\frac{R}{p} \equiv \frac{\alpha}{T}$ (ideal gas law)

$$\rightarrow \underbrace{\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right)}_{\text{local change in thickness}} = \underbrace{-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)}_{\text{geostrophic thickness advection}} + \underbrace{\omega\sigma}_{\text{adiabatic cooling/warming}} + \underbrace{\frac{\alpha}{T}\frac{1}{c_p}\frac{dQ}{dt}}_{\text{diabatic cooling/warming}}$$