The Quasi-Geostrophic Approximation ①

- difference between geostrophic approximation and quasi-geostrophic approximation:

geostrophic approximation: $\vec{V} = \vec{V_g} \longrightarrow \vec{V_{ag}} = 0$

quasi-geostrophic approximation: $\vec{V} = \vec{V_g} + \vec{V_{ag}}$, but $\underline{\vec{V_{ag}} \ll \vec{V_g}}$

Derivation of QG momentum eqns:

- use $u = u_g + u_{ag}$ and $v = v_g + v_{ag}$

$$\rightarrow \frac{d}{dt}(u_g + u_{ag}) = f(v_g + v_{ag}) - \frac{\partial \Phi}{\partial x} + F_x \qquad ①$$

$$\rightarrow \frac{d}{dt}(v_g + v_{ag}) = -f(u_g + u_{ag}) - \frac{\partial \Phi}{\partial y} + F_y \qquad ②$$

- expand $\frac{d}{dt}$ into local and advective components:

$$\rightarrow \frac{\partial}{\partial t}(u_g + u_{ag}) + (u_g + u_{ag})\frac{\partial}{\partial x}(u_g + u_{ag}) + (v_g + v_{ag})\frac{\partial}{\partial y}(u_g + u_{ag}) + w\frac{\partial}{\partial z}(u_g + u_{ag}) = f(v_g + v_{ag}) - \frac{\partial \Phi}{\partial x} + F_x \qquad ③$$

$$\rightarrow \frac{\partial}{\partial t}(v_g + v_{ag}) + (u_g + u_{ag})\frac{\partial}{\partial x}(v_g + v_{ag}) + (v_g + v_{ag})\frac{\partial}{\partial y}(v_g + v_{ag}) + w\frac{\partial}{\partial z}(v_g + v_{ag}) = -f(u_g + u_{ag}) - \frac{\partial \Phi}{\partial y} + F_y \qquad ④$$

From scale analysis of ③ and ④, the following terms are $\boxed{\text{eliminated}}$:

- $\boxed{\text{ageostrophic advection}}$: (much less than geostrophic advection)

  $u_{ag}\frac{\partial}{\partial x}(u_g+u_{ag})$, $v_{ag}\frac{\partial}{\partial y}(u_g+u_{ag})$, $u_{ag}\frac{\partial}{\partial x}(v_g+v_{ag})$, $v_{ag}\frac{\partial}{\partial y}(v_g+v_{ag})$

- $\boxed{\text{vertical advection}}$: (much less than horizontal advection)

  $w\frac{\partial}{\partial z}(u_g+u_{ag})$, $w\frac{\partial}{\partial z}(v_g+v_{ag})$

- $\boxed{\text{friction}}$: (~zero above boundary layer)

  F_x, F_y

- $\boxed{\text{local ageostrophic wind tendency}}$: ($\vec{v}_{ag} \ll \vec{v}_g$)

  $\frac{\partial u_{ag}}{\partial t}$, $\frac{\partial v_{ag}}{\partial t}$

- $\boxed{\text{advection of ageostrophic wind}}$: ($\vec{v}_{ag} \ll \vec{v}_g$)

  $u_g\frac{\partial}{\partial x}(u_{ag})$, $v_g\frac{\partial}{\partial y}(u_{ag})$, $u_g\frac{\partial}{\partial x}(v_{ag})$, $v_g\frac{\partial}{\partial y}(v_{ag})$

→ remaining terms are QG momentum eqns:

$$\frac{\partial u_g}{\partial t} = -u_g\frac{\partial u_g}{\partial x} - v_g\frac{\partial u_g}{\partial y} + f(v_g+v_{ag}) - \frac{\partial \Phi}{\partial x} \quad ⑤$$

$$\frac{\partial v_g}{\partial t} = -u_g\frac{\partial v_g}{\partial x} - v_g\frac{\partial v_g}{\partial y} - f(u_g+u_{ag}) - \frac{\partial \Phi}{\partial y} \quad ⑥$$

$\boxed{\text{- or -}}$

$$\frac{du_g}{dt} = \underbrace{fv_g - \frac{\partial \Phi}{\partial x}}_{\text{geostrophic balance}} + fv_{ag} \quad ⑦$$

$$\frac{dv_g}{dt} = \overbrace{-fu_g - \frac{\partial \Phi}{\partial y}}^{\text{geostrophic balance}} - fu_{ag} \quad ⑧$$