The Vorticity Equation

Derivation: begin with horizontal eqn of motion:

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{F}$$

- recall that relative vorticity, ζ is the $\hat{k}$ component of the curl of the 3D wind:

$$\hat{k}\cdot\vec{\omega} \equiv \hat{k}\cdot\vec{\nabla}\times\vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- so, take $\hat{k}\cdot\vec{\nabla}\times\left(\frac{d\vec{V}}{dt}\right)$ to get $\frac{d\zeta}{dt}$ eqn

① $$\hat{k}\cdot\vec{\nabla}\times\frac{d\vec{V}}{dt} = \hat{k}\cdot\vec{\nabla}\times(-2\vec{\Omega}\times\vec{V}) + \hat{k}\cdot\vec{\nabla}\times\left(-\frac{1}{\rho}\nabla p\right) + \hat{k}\cdot\vec{\nabla}\times\vec{F}$$

- expand LHS: $k\cdot\vec{\nabla}\times\frac{d\vec{V}}{dt} \equiv \frac{\partial}{\partial x}\left(\frac{dv}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{du}{dt}\right)$

② $$\hat{k}\cdot\nabla\times\frac{d\vec{V}}{dt} \equiv \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

rearranging, get

③ $$\hat{k}\cdot\nabla\times\frac{d\vec{V}}{dt} \equiv \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$$

- eqn can be written as

(4) $\hat{k}\cdot\vec{\nabla}\times\frac{d\vec{v}}{dt} \equiv \frac{\partial \zeta}{\partial t} + \vec{V}\cdot\nabla\zeta + (\nabla\cdot\vec{V})\zeta + \hat{k}\cdot\left(\nabla w \times \frac{\partial \vec{V}}{\partial z}\right)$

(HW: fill in steps between (2) and (4))

- now, deal with terms on RHS of (1)

$$\hat{k}\cdot\vec{\nabla}\times\frac{d\vec{v}}{dt} = \underbrace{\hat{k}\cdot\vec{\nabla}\times(-2\vec{\Omega}\times\vec{V})}_{A} + \underbrace{\hat{k}\cdot\vec{\nabla}\times\left(-\frac{1}{\rho}\nabla p\right)}_{B} + \underbrace{\hat{k}\cdot\vec{\nabla}\times\vec{F}}_{C}$$

(A) $\hat{k}\cdot\vec{\nabla}\times(\underbrace{-2\vec{\Omega}\times\vec{V}}_{\text{coriolis force}})$

$$-2\vec{\Omega}\times\vec{V} \equiv 2\Omega\,(v\sin\phi\,\hat{i} - u\sin\phi\,\hat{j} + u\cos\phi\,\hat{k})$$

(neglecting w contribution to coriolis force)

$$\hat{k}\cdot\vec{\nabla}\times(-2\vec{\Omega}\times\vec{V}) = 2\Omega\begin{vmatrix} 0 & 0 & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v\sin\phi & -u\sin\phi & u\cos\phi \end{vmatrix}$$

$$\equiv 2\Omega\left[\frac{\partial}{\partial x}(-u\sin\phi) - \frac{\partial}{\partial y}(v\sin\phi)\right]$$

$$= -2\Omega\frac{\partial u}{\partial x}\sin\phi - 2\Omega u\cos\phi\frac{\partial \phi}{\partial x} - 2\Omega\frac{\partial v}{\partial y}\sin\phi - 2\Omega v\cos\phi\frac{\partial \phi}{\partial y}$$

$-2\Omega \frac{\partial u}{\partial x} \sin\phi \longrightarrow -f \frac{\partial u}{\partial x}$ (recall that $f = 2\Omega \sin\phi$)

$-2\Omega u \cos\phi \cancelto{0}{\frac{\partial \phi}{\partial x}} \longrightarrow 0$

$-2\Omega \frac{\partial v}{\partial y} \sin\phi \longrightarrow -f \frac{\partial v}{\partial y}$

$-2\Omega v \cos\phi \frac{\partial \phi}{\partial y} \longrightarrow -2\Omega v \cos\phi \frac{\partial \phi}{a \partial \phi}$ ($\partial y = a\partial\phi$, where a is radius of earth)

$$\rightarrow -v \underbrace{\frac{2\Omega \cos\phi}{a}}_{\partial f/\partial y}$$

$f = f_0 + \frac{\partial f}{\partial y}(y - y_0) \rightarrow$ linear Taylor series expansion around latitude ϕ_0

$\rightarrow f_0 = 2\Omega \sin\phi_0$

$$\beta \equiv \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\underbrace{(2\Omega \sin\phi)}_{f} = 2\Omega \cos\phi \frac{\partial \phi}{\partial y}$$

$$= 2\Omega \cos\phi \frac{\partial \phi}{a \partial \phi} \rightarrow \frac{2\Omega \cos\phi}{a}$$

$\downarrow$

latitudinal variation of coriolis parameter

So term A on RHS becomes

$$-f\frac{\partial u}{\partial x} - f\frac{\partial v}{\partial y} - \beta v \longrightarrow -f(\vec{\nabla} \cdot \vec{v}) - \beta v$$

(B) $\hat{k}\cdot\vec{\nabla}\times\underbrace{\left(-\frac{1}{\rho}\nabla p\right)}_{\text{PGF}}$

$$= \hat{k}\cdot\nabla\times(-\alpha\nabla p) = \begin{vmatrix} 0 & 0 & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\alpha\frac{\partial p}{\partial x} & -\alpha\frac{\partial p}{\partial y} & -\alpha\frac{\partial p}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial x}\left(-\alpha\frac{\partial p}{\partial y}\right)\hat{\imath} - \frac{\partial}{\partial y}\left(-\alpha\frac{\partial p}{\partial x}\right)\hat{\jmath}\right]$$

$$= -\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} - \alpha\frac{\partial}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} + \alpha\frac{\partial}{\partial y}\frac{\partial p}{\partial x}$$

terms sum to zero (change order of differentiation)

$$= -\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}$$

$$\longrightarrow \hat{k}\cdot(\vec{\nabla}p\times\vec{\nabla}\alpha)$$

(C) $k\cdot\vec{\nabla}\times\vec{F} \rightarrow \hat{k}$ component of curl of friction

so all terms together:

$$\frac{\partial \zeta}{\partial t} + \vec{V}\cdot\nabla\zeta + (\nabla\cdot\vec{V})\zeta + \hat{k}\cdot\left(\nabla w\times\frac{\partial\vec{V}}{\partial z}\right) = -f(\nabla\cdot\vec{V}) - \beta v + \hat{k}\cdot\left(\nabla w\times\frac{\partial\vec{V}}{\partial x}\right) + \hat{k}\cdot(\vec{\nabla}p\times\vec{\nabla}\alpha) + \hat{k}\cdot\vec{\nabla}\times\vec{F}$$

now isolate $\frac{\partial \zeta}{\partial t}$

$$\rightarrow \frac{\partial \zeta}{\partial t} = \underbrace{-\vec{V}\cdot\nabla_z \zeta - w\frac{\partial \zeta}{\partial z}}_{A} \underbrace{-\beta v}_{B} \underbrace{-(\zeta+f)(\nabla\cdot\vec{V})}_{C}$$

$$\underbrace{-\hat{k}\cdot\left(\nabla w \times \frac{\partial \vec{V}}{\partial z}\right)}_{D} + \underbrace{\hat{k}\cdot(\nabla p \times \nabla \alpha)}_{E} + \underbrace{\hat{k}\cdot\vec{\nabla}\times\vec{F}}_{F}$$

Eularian form of vorticity equation

Physical Interpretation of Terms

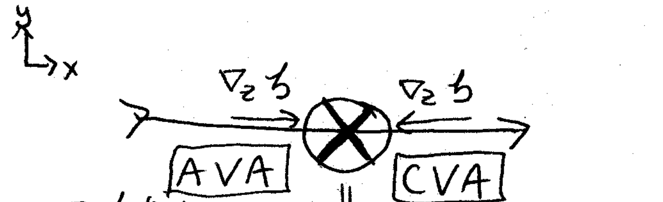
(A) $\frac{\partial \zeta}{\partial t} \propto \underset{①}{-\vec{V}\cdot\nabla_z \zeta} \underset{②}{- w\frac{\partial \zeta}{\partial z}}$

① horizontal advection of vorticity
② vertical advection of vorticity

①

$-\vec{V}\cdot\nabla_z \zeta > 0 \rightarrow$ cyclonic vorticity advection

$-\vec{V}\cdot\nabla_z \zeta < 0 \rightarrow$ anticyclonic vorticity advection



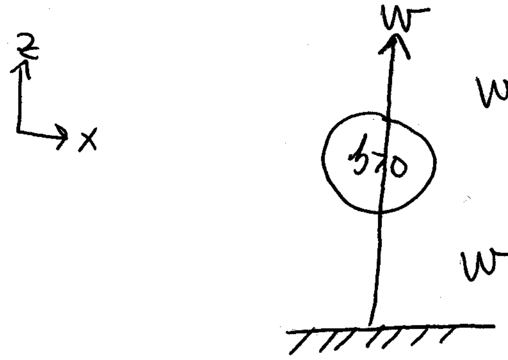
$\longrightarrow$ $\vec{V}$

$\otimes$ vorticity max

AVA	CVA
$\partial\zeta/\partial t < 0$	$\partial\zeta/\partial t > 0$
$\vec{V}$ and $\nabla_z \zeta$ point in same direction	$\vec{V}$ and $\nabla_z \zeta$ point in opposite directions

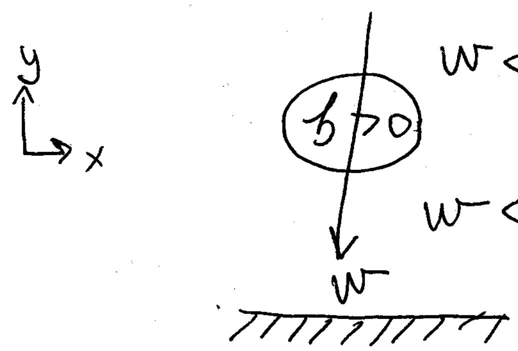
② $-w\frac{\partial \zeta}{\partial z} > 0 \rightarrow$ cyclonic vertical vorticity advection

$-w\frac{\partial \zeta}{\partial z} < 0 \rightarrow$ anticyclonic vertical vorticity advection



$w > 0, \; \frac{\partial \zeta}{\partial z} < 0 \rightarrow \frac{\partial \zeta}{\partial t} > 0$

$w > 0, \; \frac{\partial \zeta}{\partial z} > 0 \rightarrow \frac{\partial \zeta}{\partial t} < 0$



$w < 0, \; \frac{\partial \zeta}{\partial z} < 0 \rightarrow \frac{\partial \zeta}{\partial t} < 0$

$w < 0, \; \frac{\partial \zeta}{\partial z} > 0 \rightarrow \frac{\partial \zeta}{\partial t} > 0$

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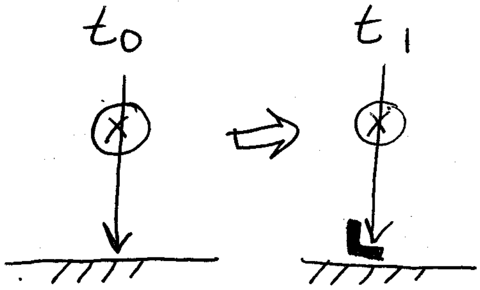
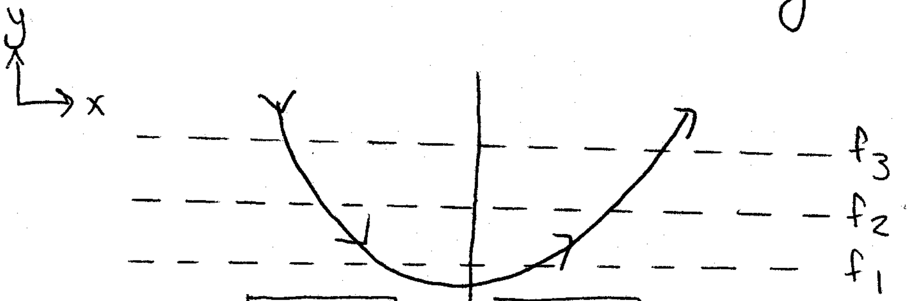
note that downdrafts ($w < 0$) can advect midlevel cyclonic vorticity down to the surface



B $\frac{\partial \zeta}{\partial t} \propto -\beta v$

$\propto -v \frac{\partial f}{\partial y} \rightarrow$ meridional advection of planetary vorticity



$f_3 > f_2 > f_1$

$\frac{\partial f}{\partial y} > 0$ [f increases to N]

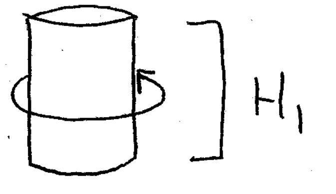
$\boxed{v<0}$	$\boxed{v>0}$
$\frac{\partial \zeta}{\partial t} > 0$	$\frac{\partial \zeta}{\partial t} < 0$
cyclonic planetary vort. adv on west side of trough	anticyclonic planetary vort. advection on east side of trough

· this term explains why a cutoff low (with little relative vorticity advection) will tend to drift westward

C $\frac{\partial \zeta}{\partial t} \propto -(\zeta+f)(\nabla \cdot \vec{V}) \longrightarrow -\zeta(\nabla \cdot \vec{V})_{①} - f(\nabla \cdot \vec{V})_{②}$

· divergence term

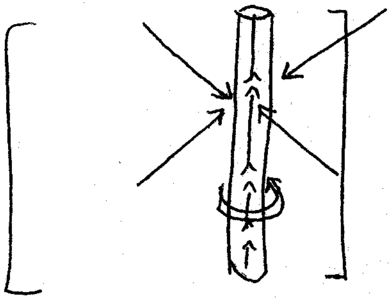
① consider a column with preexisting cyclonic relative vorticity



t_0

- add convergence ($\nabla \cdot \vec{V} < 0$)

$\frac{\partial w}{\partial z} > 0$



H_2

t_1

$H_1 < H_2$

→ column spins faster since it is stretched

- fluid analog to conservation of angular momentum in rigid-body mechanics
  (ice skater pulls in arms to spin faster)

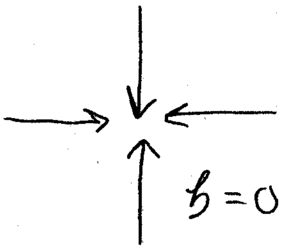
- remember that $-\nabla \cdot \vec{V} = \frac{\partial w}{\partial z}$ for incompressible fluid
  (continuity equation / conservation of mass)

CON: $\nabla \cdot \vec{V} < 0 \rightarrow \frac{\partial w}{\partial z} > 0 \rightarrow$ stretching

DIV: $\nabla \cdot \vec{V} > 0 \rightarrow \frac{\partial w}{\partial z} < 0 \rightarrow$ shrinking

② $-f(\nabla \cdot \vec{V})$

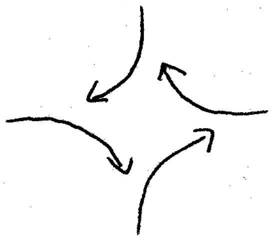
convergent scenario



$\zeta = 0$

t_0

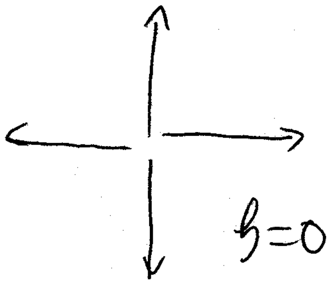
coriolis force means that converging air acquires cyclonic relative vorticity



$\zeta > 0$

t_1

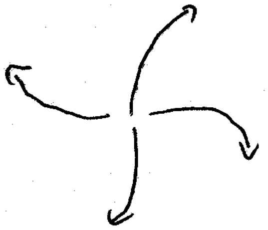
divergent scenario



$\zeta = 0$

t_0

c.f → diverging air acquires anticyclonic relative vorticity



$\zeta < 0$

CON: $\nabla \cdot \vec{V} < 0 \rightarrow \frac{\partial \zeta}{\partial t} > 0$

DIV: $\nabla \cdot \vec{V} > 0 \rightarrow \frac{\partial \zeta}{\partial t} < 0$

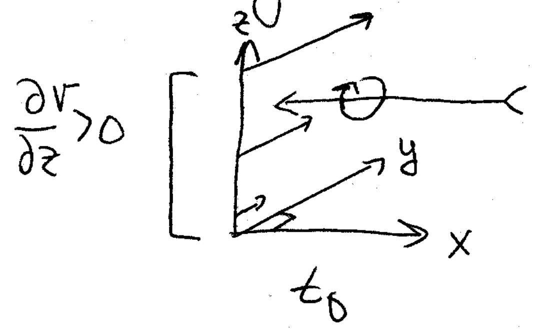
D $\frac{\partial \zeta}{\partial t} \propto -\hat{k} \cdot \left(\nabla w \times \frac{\partial \vec{V}}{\partial z} \right)$

- tilting term
- horizontal vorticity (vertical wind shear) is tilted into the vertical by a horizonta gradient of w

$$\frac{\partial \zeta}{\partial t} \propto - \underbrace{\frac{\partial w}{\partial x} \frac{\partial v}{\partial z}}_{①} + \underbrace{\frac{\partial w}{\partial y} \frac{\partial u}{\partial z}}_{②}$$

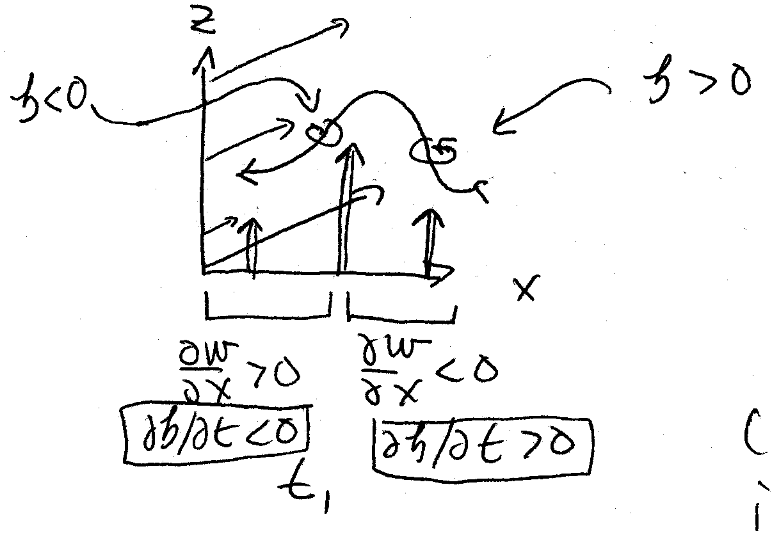
- consider 1st term

· have wind becoming increasingly southerly with height:



vertical wind shear creates horizontal vortex tube

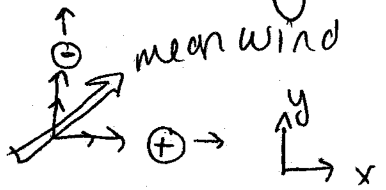
· add horizontal gradient of w $\left(\frac{\partial w}{\partial x}\right)$:



tilting of vortex tube creates cyclonic or anticyclonic relative vorticity (horizontal vorticity is tilted into vertical vorticity)

· argument works the same way for 2nd term $\left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$

→ how you get supercell splitting (a cyclonic and anticyclonic rotating cell)



⊕ cyclonic supercell
⊖ anticyclonic s. cell

(E) $\frac{\partial \zeta}{\partial t} \propto \hat{k} \cdot (\nabla p \times \nabla \alpha) \rightarrow$ solenoid term

when ∇p not parallel to $\nabla \alpha$, can get cyclonic or anticyclonic spin up

(F) $\frac{\partial \zeta}{\partial t} \propto \hat{k} \cdot \vec{\nabla} \times \vec{F}$

- friction in PBL acts to spin down both cyclones and anticyclones through Ekman pumping

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<u>Conservation of Potential Vorticity</u>

- can simplify vorticity eqn using geostrophic assumptions and derive

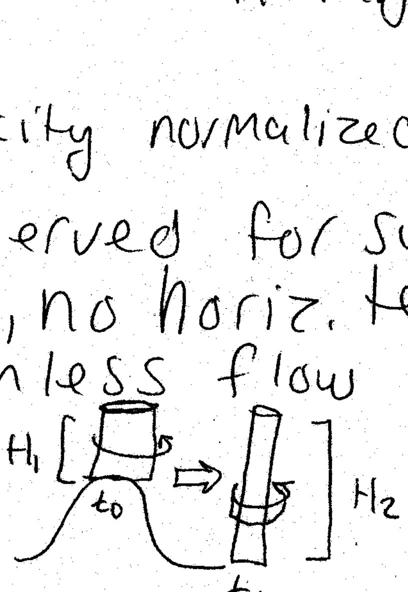
$$\frac{d}{dt}\underbrace{\left(\frac{\zeta_g + f}{H}\right)}_{\text{Potential vorticity}} = 0$$

ζ_g: geostrophic rel. vorticity
f: planetary vorticity
H: height of column

↳ absolute vorticity normalized by column height

- says that PV is conserved for synoptic-scale, barotropic (i.e., no horiz. temp. gradients) incompressible, frictionless flow

$\boxed{\begin{array}{l}\text{if } H \uparrow, \ \zeta + f \text{ must } \uparrow \\ \text{if } H \downarrow, \ \zeta + f \text{ must } \downarrow\end{array}}$



$H_2 > H_1$, so $(\zeta_g + f)_2 > (\zeta_g + f)_1$